Exam 1

This exam covers topics from chapters 6 and 7 in Rogawski. Unless stated explicitly in the problem, **you must show all work for full credit**. Put a box around your final answers. You will have 80 minutes for the exam. Class will resume again at 7:30pm, so you will have at least 10 minutes for a break. If you finish the exam early, then please take your break outside the classroom. There are 110 points available for this exam, but the maximum score possible is 100. It counts for 20 percent of your final grade.

By signing below, you promise that you will not cheat in any way on this exam.

Name:

Signature:

Problem	**Points Possible**	**Points Earned**
1a	10	
1b	10	
1c	10	
1d	10	
2a	10	
2b	10	
3a	10	
3b	10	
4a	10	
4b	10	
5	10	
Grade:		

1 Integration Time

Compute the following integrals. Use whatever method you like, but make sure to state what methods you use when you use them.

(a) $\int \ln^3 x \, dx$ For your convenience: $\int \ln x \, dx = x \ln x - x + C$.
Using integration by parts:

$$u = \ln^3 x \qquad v = x$$
$$du = \frac{3 \ln^2 x}{x} \, dx \qquad dv = 1 \, dx$$

We obtain

$$\int \ln^3 x \, dx = x \ln^3 x - 3 \int \ln^2 x \, dx$$

We then use integration by parts again:

$$u = \ln^2 x \qquad v = x$$
$$du = \frac{2 \ln x}{x} \, dx \qquad dv = 1 \, dx$$

We then get

$$\begin{aligned} x \ln^3 x - 3 \int \ln^2 x \, dx &= x \ln^3 x - 3 \left(x \ln^2 x - \int \ln x \, dx \right) \\ &= x \ln^3 x - 3 \left(x \ln^2 x - (x \ln x - x) \right) + C \end{aligned}$$

(b) $\int \frac{14x - 26}{(3x-7)(x+1)} \, dx$
The partial fraction decomposition is

$$\frac{14x - 26}{(3x-7)(x+1)} = \frac{4}{x+1} + \frac{2}{3x-7}$$

Therefore, we have

$$\int \frac{14x - 26}{(3x-7)(x+1)} \, dx = \int \frac{4}{x+1} + \frac{2}{3x-7} \, dx = 4 \ln |x+1| + \frac{2}{3} \ln |3x-7| + C$$

(c) $\int \sin^3 x \cos^5 x \; dx$
We isolate one $\sin x$:

$$\begin{aligned} \int \sin^3 x \cos^5 x \; dx &= \int \sin x (1 - \cos^2 x) \cos^5 x \; dx \\ &= \int \sin x (\cos^5 x - \cos^7 x) \; dx \end{aligned}$$

If we make the substitution $u = \cos x$, then $\frac{du}{dx} = -\sin x$ so $dx = -\frac{du}{\sin x}$ and

$$\begin{aligned} \int \sin x (\cos^5 x - \cos^7 x) \; dx &= \int \sin x (u^5 - u^7) \frac{-du}{\sin x} \\ &= \int u^7 - u^5 \; du \\ &= \frac{u^8}{8} - \frac{u^6}{6} + C \\ &= \frac{\cos^8 x}{8} - \frac{\cos^6 x}{6} + C \end{aligned}$$

(d) $\int \frac{t}{\sqrt{t^2 - 16}} \; dt$
We make the u-substitution $u = t^2 - 16$, yielding $\frac{du}{dt} = 2t$ so $dt = \frac{du}{2t}$. Therefore,

$$\begin{aligned} \int \frac{t}{\sqrt{t^2 - 16}} \; dt &= \int \frac{t}{\sqrt{u}} \frac{du}{2t} \\ &= \frac{1}{2} \int \frac{1}{\sqrt{u}} \; du \\ &= \frac{1}{2} 2\sqrt{u} + C \\ &= \sqrt{t^2 - 16} \end{aligned}$$

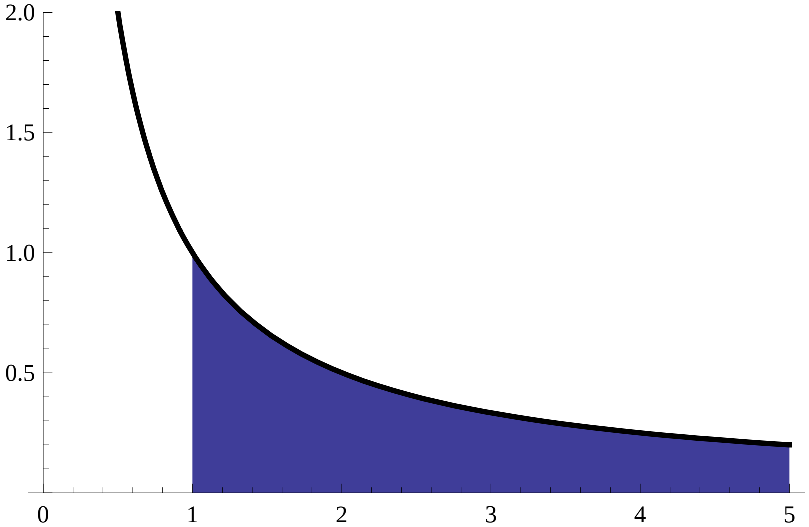
2 Gabriel's Horn

Gabriel's Horn is the mathematical object created by rotating the area bounded by the the curves

$$y = \frac{1}{x}$$
$$y = 0$$
$$x = 1$$

around the x-axis.

(a) Show that the "long" cross-section of Gabriel's Horn is infinite. For your convenience, a partial sketch is shown, but remember that the bounds of the integral!



$$\int_1^\infty \frac{1}{x}\,dx = (\ln x + C)|_1^\infty = \ln\infty - \ln 1 = \infty$$

(b) Show (by computing the appropriate integral) that the volume of Gabriel's Horn is finite. Weird, huh? At some arbitrary x-value, the cross-section is a disk with radius $\frac{1}{x}$ and area $\frac{\pi}{x^2}$. Therefore, the volume is

$$\int_1^\infty \frac{\pi}{x^2}\,dx = \pi\int_1^\infty \frac{1}{x^2}\,dx$$

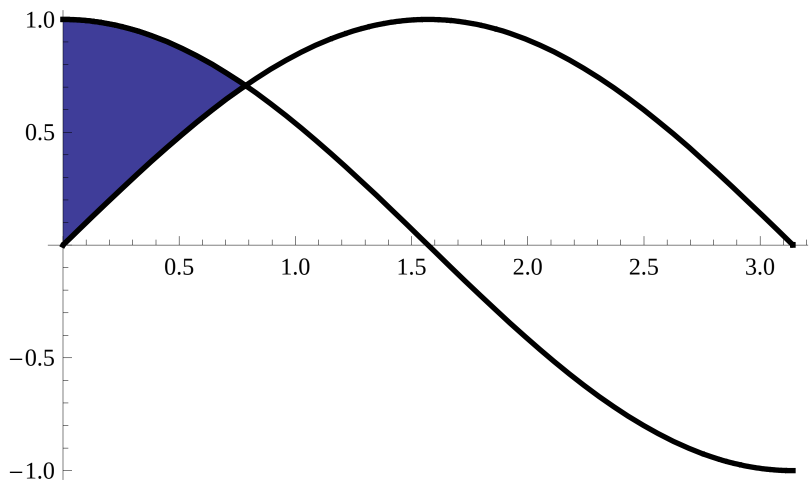
which is finite since $\int_1^\infty \frac{1}{x^2}\,dx$ is finite (and, specifically, is equal to 1).

3 Volumes.

Let A be the area bounded by the curves

$$\begin{aligned} y &= \sin x \\ y &= \cos x \\ x &= 0 \end{aligned}$$

For your convenience, a sketch is shown:



(a) Set up, *but do not evaluate*, the integral representing the volume of the solid obtained by rotating A around the x-axis.
The intersection point of these two functions is $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$. At some arbitrary x-value, our cross-section is a washer with inner radius $\sin x$ and outer radius $\cos x$. Therefore, the cross-sectional area is $\pi(\cos^2 x - \sin^2 x)$ and the volume is

$$\int_0^{\frac{\pi}{4}} \pi(\cos^2 x - \sin^2 x)\ dx = \pi \int_0^{\frac{\pi}{4}} \cos^2 x - \sin^2 x\ dx$$

(b) Set up, *but do not evaluate*, the integral representing the volume of the solid obtained by rotating A around the y-axis.
At some arbitrary x-value, our cross-section is a cylinder with radius x and height $\cos x - \sin x$, so the volume is

$$\int_0^{\frac{\pi}{4}} 2\pi x(\cos x - \sin x)\ dx$$

4 Trapezoid Rule

Approximate the following definite integral using the Trapezoidal Rule with $N = 4$:

$$\int_1^5 x^2 - 6x + 5 \, dx$$

For your convenience, here is a table of values:

x	$x^2 - 6x + 5$
0	5
1/2	9/4
1	0
3/2	-7/4
2	-3
5/2	-15/4
3	-4
7/2	-15/4
4	-3
9/2	-7/4
5	0

Following the Trapezoid Rule, we have

$$\begin{aligned} \int_1^5 x^2 - 6x + 5 \, dx &= \frac{1}{2}\left(f(1) + 2f(2) + 2f(3) + 2f(4) + f(5)\right) \\ &= \frac{1}{2}(0 - 6 - 10 - 6 + 0) \\ &= -10 \end{aligned}$$

(b) Using the error bounds for the Trapezoidal Rule, show that the estimate you got in part (a) was indeed within the bounds of the actual definite integral. (*Hint:* Keep everything in fractions - don't use decimals.) If $f(x) = x^2 - 6x + 5$ then $f''(x) = 2$ so $K_2 = 2$. Hence,

$$|Error(T_4)| \leq \frac{K_2(b-a)^3}{12(4)^2} = \frac{2(4)^3}{12(4)^2} = \frac{8}{12} = \frac{2}{3}$$

The actual integral is

$$\begin{aligned} \int_1^5 x^2 - 6x + 5 \, dx &= \left(\frac{x^3}{3} - 3x^2 + 5x + C\right)\Big|_1^5 \\ &= \left(\frac{5^3}{3} - 3(5)^2 + 5(5)\right) - \left(\frac{1^3}{3} - 3(1)^2 + 5(1)\right) \\ &= \left(\frac{125}{3} - \frac{225}{3} + \frac{75}{3}\right) - \left(\frac{1}{3} - \frac{9}{3} + \frac{15}{3}\right) \\ &= \frac{-25}{3} - \frac{7}{3} = \frac{-32}{3} \end{aligned}$$

which is *precisely* the maximum amount of error we could have!

5 Work It

Calculate the work done against gravity draining a pool from the top that has a rectangular base that is 5 meters wide and 10 meters long and is 2 meters tall. Assume that the pump is directly at the top of the pool. Remember that the density of water is $1000kg/m^3$ and force is given by $F = mg$, where $g = 9.8m/s^2$. It is probably best to not simplify your answer. At some level y in the pool, a slice of the pool with height Δy has volume $50\Delta y$ cubic meters, and hence has mass $50000\Delta y$ kilograms. This slice needs to travel $2 - y$ to be drained, so the total amount of work needed for this slice is $50000(9.8)\Delta y(2 - y)$. Therefore, the total amount of work needed is

$$\begin{aligned}
\int_0^2 50000(9.8)y(2-y)\,dy &= 50000(9.8)\int_0^2 2y - y^2\,dy \\
&= 50000(9.8)\left(y^2 - \frac{y^3}{3}\right)\Bigg|_0^2 \\
&= 50000(9.8)\left[\left(2^2 - \frac{2^3}{3}\right) - \left(0^2 - \frac{0^3}{3}\right)\right] \\
&= 50000(9.8)\left(\frac{4}{3} - 0\right) \\
&= 50000(9.8)\frac{4}{3}
\end{aligned}$$