The number of tetrahedra with minimal, uniform, and distinct volumes in $3D$

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Abstract

Determining the maximum number of unit distances determined by n points in the plane is one of the notoriously hard Erdős problems in combinatorial geometry. It is easy to give a tight bound on number of occurrences of the minimum and maximum distance, which is at most $3n - o(n)$ and n, respectively. Finding the maximum number of unit area triangles determined by n points in the plane is similarly hard as the unit distance problem. It is known, however, that the minimum and maximum triangle areas can occur $O(n^2)$ and $O(n)$ times, and both bounds are tight. We pursue the analogous problems in the space, and find bounds on the maximal number of unit, minimum, and maximum volume tetrahedra determined by n points in three dimensions, along with some new techniques.

(i) The number of minimum (positive) volume tetrahedra spanned by n points in $3D$ is $O(n^3 \log^2 n)$, and there are point sets for which this number is $\Omega(n^3)$. (ii) The number of unit-volume tetrahedra determined by n points in $3D$ is $O(n^{7/2})$, and there are point sets for which this number is $\Omega(n^3 \log\log n)$. (iii) The tetrahedra determined by n points in $3D$, not all on a plane, have at least $\Omega(n/\sqrt{\log n})$ distinct volumes, and there are point sets for which this number is $O(n)$. (Joint work with Adrian Dumitrescu)

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