# 642:587 Convex and Discrete Geometry, Homework 1

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**Homework 1 must be handed in class on Wednesday, 09/21/05**

**You are required to hand in written solutions for 5 problems.**

**Problems from the book**

Section 3.1: $1, 2, 4, 5$.

Section 3.2: $2, 5, 6$.

**Problem:** Prove that the number of empty convex quadrilaterals in the $n$-point Horton set is at most $3n^2$.

**Problem:** Given a sufficiently large set of points in the plane in general position, connect every two points with a straight-line segment, and color each segment red or blue. Show that there exists an empty monochromatic triangle.

**Problem: Alternative proof of $g_3(n) \leq 2n^2$.** Let $I_i$, $i = 1, \ldots, n$, be the vertical unit segments:

$$I_i = \{(x, y) : x = i, 0 \leq y \leq 1\}.$$

Choose a point $p_i$ in $I_i$ randomly with uniform distribution. Show that the expected number of empty triangles determined by the points $p_i$, $i = 1, \ldots, n$, is at most $2n^2$.

**Problem:** Given a set of $n$ points in the plane in general position, show that the number of convex empty quadrilaterals is at least $\frac{1}{2}n^2 - O(n \log n)$. Proving a $(\frac{1}{2} + c)n^2$ lower bound for some constant $c > 0$ is an open problem.

**Problem:** Using the continuous motion argument from class, prove that

$$\sum_{k \geq 3} (-1)^{k+1} k X_k(P) = 2\binom{n}{2} - H(P),$$

where $P$ is a set of $n$ points in the plane (in gen. pos.), $H(P)$ is the number of edges of the convex hull of $P$, and $X_k(P)$, $k \geq 3$, denotes the number of empty convex $k$-gons determined by the points of $P$.

**Problem:** Let $X \cup Y$ be a set of points in gen. pos. in the plane with no 6-hole. Suppose that $X$, $Y$ are both in convex position and each point of $Y$ lies inside $\text{conv}(X)$. Prove that $|Y| \leq 5$ and $|X| \leq 7$. Can you use this to improve the bound given by Valtr's proof of the "empty convex hexagon theorem", presented in class?

**Problem:** Can you color the points of the Horton set with three colors so that there is no monochromatic empty triangle?

**Open Problem:** Is it true that every sufficiently large 2-colored set of points in general position in the plane contains vertices of a monochromatic empty convex quadrilateral?