

642:587 Convex and Discrete Geometry, Homework 2

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Homework 2 must be handed in class on Wednesday, 10/05/05

You are required to hand in written solutions for 6 problems. Problem 6 counts as two.

Problem 1: Find all planar point configurations with n points that determine exactly n different lines, $n \geq 4$.

Problem 2: (a) For every n , $n \geq 6$, find n points in \mathbb{R}^3 such that the plane determined by any three noncollinear points contains at least four points.

(b) Find a finite point set in the complex plane \mathbb{C}^2 that does not lie in a complex line and the complex line determined by any two points contains at least three points.

Problem 3: (a) Prove the following: Let P be a finite set of points in \mathbb{R}^2 . If any three points of P determine a circle that passes through a fourth point of P , then all points of P lie on a circle. (hint: Think of a geometric transformation that sends circles into lines.)

(b) Prove the following: Let P be a finite set of points in \mathbb{R}^3 , not all in one plane. Show that there exists a plane spanned by the points of P , such that all the points of P in this plane lie on two straight lines intersecting in one of the points.

Problem 4: Prove the following: Let P be a finite set of points in \mathbb{R}^3 , not all in one plane. Show that there exists a plane spanned by the points of P , such that all but exactly one of the points of P in this plane are contained in a line. (Note: Hansen proved the existence of at least $\frac{2n}{5}$ such planes. It is an open problem to improve this bound.)

Problem 5: (a) While proving a theorem of Pach and Pinchasi on bichromatic lines, we used the following easy corollary of Euler's theorem:

$$\sum_{i=2}^{n-1} (i-3)t_i \leq -3,$$

where t_i is the number of connecting lines through exactly i points of a non-collinear point set P in \mathbb{R}^2 . Suppose n is even. Use this inequality to show that $t_2 > \frac{n}{6}$.

- (b) Find 7 points in the plane spanning exactly 3 ordinary lines.
- (c) Find 13 points in the plane spanning exactly 6 ordinary lines.

Problem 6: Kelly-Moser proof of the existence of $3n/7$ ordinary lines Let P be a finite set of points in \mathbb{R}^2 , not all collinear. Let S be the set of connecting lines. Given $p \in P$, the set of lines of S which do not go through p subdivide the plane into polygonal regions. The region containing p is called the *residence* of p . The lines of S containing the edges of the residence of p are the *neighbours* of p . The number of ordinary lines through p is the *order* of p . The number of neighbours of p which are ordinary lines is the *rank* of p . The sum of the order and the rank of p is the *index* of p .

Prove the following:

1. If a point has precisely one neighbour, then S is a near-pencil (a collection of lines, all but one of which intersect at a single point).
2. If a point has precisely two neighbors, then S is a near-pencil.
3. If the order of p is zero, then every neighbor of p is an ordinary line.
4. The index of each point of P which is not of order two is at least three.
5. If a line s of S is a neighbor of three points p_1, p_2, p_3 , then the points of P which lie on s are on the connecting lines determined by p_1, p_2, p_3 .
6. A line of S is a neighbor of at most four points.
7. If I_i is the index of the point p_i , then

$$m \geq \frac{1}{6} \sum_{i=1}^n I_i,$$

where m is the number of ordinary lines.

8. Finish the proof, i.e. show that $m \geq 3n/7$.

Problem 7: (a) Find four blue and four red points in the plane, not all on a line, such that there is no bichromatic ordinary line, although the red points cannot be separated from the blue ones by a straight line,

(b) Given n red and n blue points in the plane, not all on a line, prove that the number of bichromatic lines passing through at most 6 points is at least $\frac{1}{10}$ of the total number of connecting lines.

(c) Find six red and three blue points in the plane, not all on a line, such that there is no bichromatic ordinary line, although the red points can be separated from the blue ones by a straight line.

Problem 8: Given a set of n points in \mathbb{R}^3 , not all in one plane, and no three on a line, show that there is a direction spanned by at most $\frac{n}{4}$ different pairs of points.

(Note: Open problem: Improve $\frac{n}{4}$ above!)

Problem 9: (a) Let \mathcal{C} be a finite family of at least two pairwise intersecting unit circles in the plane. Prove that there exists an intersection point through which at most five circles pass.

(b) Find a family of 7 pairwise intersecting circles (not all of the same radius!) such that there exists no intersection point through which at most two circles pass.

Problem 10: How many distinct slopes are determined by the point set $\{(a, b) \in \mathbb{N}^2 : 1 \leq a, b \leq n\}$? How about the 3-D $n \times n \times n$ integer lattice section?

Problem 11: Consider the allowable sequence of order n , that starts with $12 \dots n$. Show that in a half-period, n will switch at each of the first k positions exactly once. Conclude that the number of $\leq k$ -sets on n points in \mathbb{R}^2 is at most $2nk - 2k^2 - k$, for $n > 2k$.

(Note: A switch in position i is a switch between the i th and the $(i + 1)$ st index. An $\leq k$ -set is an i -set for some $i \leq k$.)

Problem 12: Consider the allowable sequence of order n . Show that in one period, every element occurs in at most $4k$ switches in positions $\{1, \dots, k\} \cup \{n - k, \dots, n - 1\}$. Conclude that the number of $\leq k$ -sets on n points in \mathbb{R}^2 is at most kn (and thus exactly kn).

Problem 13: Let $a(P)$ be the minimum number of angles determined by ordered triples of a set P of non-collinear points in the plane. (Count 0-angles, but not 180-angles.) Let $a(n)$ be the maximum of $a(P)$ over all n -point sets P . For every $n \geq 3$, find a set P_n of n points such that $a(P_n) = n - 2$. Show that $a(n) = \Omega(n)$. (Note: Open Problem: Prove or disprove $a(n) = n - 2$.)

Problem 14: Given a set P of n points in the plane, not all on a line, show that there is a connected graph drawn in the plane with straight line edges such that its vertex set is P and its edges have pairwise different slopes.

Problem 15: Given a set P of n points in the plane, no three on a line, and an abstract graph G with n vertices. Is there a straight line embedding of G into the plane such that the vertices of G are mapped onto P and the edges of G are pairwise nonparallel if:

- G is a path and P forms a regular n -gon.
- G is a path (Do not try this at home.)
- G is a tree and P is in convex position (open problem)
- G is a tree ?(open problem)

Problem 16: Prove that a set of $2n$ points in the plane, no three on a line, admits a matching by pairwise crossing segments if and only if it spans precisely n halving lines (i.e. lines whose both open halfspaces contain exactly $n - 1$ points each).