642:587 Convex and Discrete Geometry, Homework 3

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Homework 3 must be handed in class on Wednesday, 10/17/05

You are required to hand in written solutions for 6 problems.

Problems from the book

Section 4.1: $3, 5, 6$.

Section 4.3: $1, 2, 3$.

Section 4.4: $1, 2$.

Section 4.6: $3, 4$.

Problem: (a) Prove that there exist a positive constant c and a natural number n_0 such that any graph G with $n \geq n_0$ vertices and more than $\lceil n^2/4 \rceil$ edges contains cn triangles sharing an edge.

(b) Use (a) and prove the following statement: Given $t > 0$ and a set of n points $p_1, \ldots, p_n$ in the plane with minimum distance at least 1, the number of pairs $p_i p_j$, $i < j$, whose distance is between t and $t+1$, cannot exceed $\lceil n^2/4 \rceil$, provided that $n \geq n_0$. Furthermore, this bound can be attained for all n and for all sufficiently large $t > t_0(n)$.

Problem: (a) Let P be a set of infinitely many points in the plane such that all distinct distances determined by them are integers. Prove that all points of P are collinear.

(b) Find a set of infinitely many points in the plane, not all on a line, such that all distances determined by them are rational numbers.

(c) Show that for any $n \geq 3$, there exist n points in the plane, not all on a line, such that all distances determined by them are integers.

Problem: Show that if $\{p_1, p_2, p_3\}$, $\{q_1, q_2, q_3\}$ are two triples of points in $\mathbb{R}^d$ such that all nine pairwise distances are unit distances, then the p_i's induce a circle C and the q_j's induce a circle C' such that the 2-dimensional planes containing C and C' are orthogonal.

Problem: Let $\mathcal{L} = \{l_1, \ldots, l_n\}$ be a set of n lines in $\mathbb{R}^2$, let $w_i \geq 0$ $(1 \leq i \leq n)$ be a weight assigned to line l_i, and let $r \leq n$ be an integer. Show that one can partition the plane into $s \leq r^2$ trapezoids, so that for each trapezoid, the total weight of lines intersecting its interior is at most $c \sum_{i=1}^{n} w_i / r$, where c is a suitable constant.

Problem: Show that for any set of n points in the plane, the number of triples that determine an isosceles triangle is $O(n^{7/3})$.

Problem: Let P be a set of n points in the plane, no three of which are collinear. Prove that there exists a $p \in P$ such that the number of distinct distances from p is at least $\lceil (n-1)/3 \rceil$. (Note: Can you prove a stronger bound?)

Problem: Given n points in the plane, no 3 collinear, draw a circle through each triple. How many different circles can have unit radius? Show by construction that this number may exceed $cn^{3/2}$. (Note: Open problem: Can you show a $o(n^2)$ upper bound?)

Problem: Provide a construction for the matching upper bound in the "number-theoretic" lemma of Solymosi and C. Tóth, presented in class.

Problem: Prove that the maximum number of times the unit distance can occur among n points in $\mathbb{R}^3$ is $\Omega(n^{4/3})$ and $O(n^{5/3})$. (Hint: For the upper bound, consider an extremal graph theory argument with $K_{3,3}$.)

Problem: (a) Let X and Y be two sets of n real numbers. Consider the Cartesian product

$$P = X \times Y = \{(x, y) \in \mathbb{R}^2 : x \in X, y \in Y\}$$

in the plane. Show that the number of collinear triples of P is at most $O(n^4 \log n)$.

(b) Let A denote a finite set of nonzero real numbers. Use (a) and prove that if $|A| = n$, then $(|A + A| + n)^4 \cdot |A \cdot A| \geq \frac{cn^6}{\log n}$. Conclude that if $|A + A| \leq Cn$, then $|A \cdot A| \geq \frac{cn^2}{\log n}$. (Hint: Look at the hyperbolae covering $A \times A$, and form appropriate collinear triples in $((A + A) \cup A) \times ((A + A) \cup A)$.)