Math 250, Homework 7, Due 04/5/05

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March 25, 2005

Problem 0 Do the following problems from the book:

$$\text{Section 4.2} : 13, 17, 27, 34, 38; \ \text{Section 4.3} : 16, 31, 44;$$

$$\text{Section 5.1} : 40, 44, 48; \ \text{Section 5.2} : 2, 4, 16, 27, 46, 48, 52.$$

Problem 1 Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for $\mathbb{R}^n$. Let A be an invertible matrix. Show that $\{Av_1, Av_2, \ldots, Av_n\}$ is also a basis for $\mathbb{R}^n$.

Problem 2 Let A and C be square matrices. Find the spectrum of the matrix

$$T = \begin{pmatrix} A & B \\ O & C \end{pmatrix}$$

in terms of the spectrums of A and C.

Problem 3 Suppose that $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ are the eigenvalues for the $n \times n$ matrix A, and let (λ_k, c) be a particular eigenpair.

1. For $\lambda \notin \sigma(A)$, explain why $(A - \lambda I)^{-1} c = c/(\lambda_k - \lambda)$.
2. For an arbitrary $n \times 1$ vector d, prove that the eigenvalues of $A + cd^T$ agree with those of A except that λ_k is replaced by $\lambda_k + d^T c$. (Hint: Use the "rank one update" determinant formula from class.)

Problem 4 Given a set of n points $S = \{(x_1, y_1), \ldots, (x_n, y_n)\}$ in the plane, where x_i's are all distinct, explain why there is a unique polynomial $P(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \ldots + \alpha_{n-1} t^{n-1}$ of degree $n - 1$ that passes through each point of S.

Bonus Problem 1: Can replace 5 other problems Let A be an $m \times n$ matrix, and let B be an $n \times p$ matrix. Prove that $\text{rank}(AB) = \text{rank}(B) - \dim(\text{Null}(A) \cap \text{Col}(B))$. (Hint: Start with a basis for $\text{Null}(A) \cap \text{Col}(B)$ and extend it to the basis for $\text{Col}(B)$.) Finally, use this identity to show that $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$.

Bonus problem 2: Can replace 4 other problems: Let a and b be two real numbers. Consider an $n \times n$ matrix such that all the entries on the main diagonal are equal to b, all the entries on the diagonals immediately above and below the main diagonal are equal to a; and all the remaining entries are zero. Find the eigenvalues and eigenvectors of this matrix.