Math 250, Practice problems for Exam 2

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Problem 1: Let

$$A = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$

Verify that $(1,1,1)^T$ is an eigenvector of A. Find the characteristic polynomial of A. Find its eigenvalues. Find the corresponding eigenvectors. Find the matrix P such that $P^{-1}AP$ is a diagonal matrix. What are dimensions of its eigenspaces?

Problem 2: Given matrices

$$A = \begin{pmatrix} 1 & -2 & -1 \\ 2 & -1 & 1 \\ 7 & -8 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & -2 & -1 \\ 2 & -1 & 3 \\ 7 & -8 & 3 \\ 5 & -7 & 0 \end{pmatrix}$$

Find the basis for their row spaces, their column spaces, their nullspaces.

Problem 3: Show that the set W of all 2×2 matrices of the form

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

is a vector space. Find a basis of W. What is the dimension of W?

Problem 4: Let A be a 3×3 matrix whose eigenvalues are 0, 1, 2. Show that the matrix $A^2 + I$ has eigenvalues 1, 2, 5. Explain why $\det(A) = 0$ but $\det(A^2 + I) = 10$.

Problem 5: Evaluate the determinants of the following matrices:

$$A = \begin{pmatrix} 1 & 2 & 4 & 3 \\ -1 & 2 & 3 & -1 \\ 3 & 0 & -3 & 0 \\ 2 & 0 & 0 & -2 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 & 4 & 8 \\ 1 & 4 & 16 & 64 \\ 1 & 8 & 64 & 512 \\ 1 & 16 & 256 & 4096 \end{pmatrix}$$

Problem 6: Given that the 4×4 matrix $A = [c_1|c_2|c_3|c_4]$ has determinant 3, find the determinant of the matrix $B = [c_2 + c_3|c_3 + c_4|c_4 + c_1|c_1 + c_2]$.

Problem 7: For the system of linear equations $x+y+z = 1$, $x+2y+3z = 2$, $x+4y+9z = 4$, use Cramer's rule to solve for y alone.

Problem 8: Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for $\mathbb{R}^n$. Let A be an invertible matrix. Show that $\{Av_1, Av_2, \ldots, Av_n\}$ is also a basis for $\mathbb{R}^n$.

Problem 9:Bonus Without computing A, find a basis for $\text{Col}(A^T)$, $\text{Col}(A)$, and $\text{Null}(A)$, where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 8 & 3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Problem 10: For what values of c is the matrix not invertible?

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 3 & -2 & -c \\ 0 & c & -10 \end{pmatrix}$$

Problem 11: Suppose that A is a 4×4 matrix with no complex eigenvalues and exactly two real eigenvalues: 5 and -9. Let W_1 and W_2 be the corresponding eigenspaces. Write all the possible characteristic polynomials of A that are consistent with the following information: i) dim $W_1 = 3$, ii) dim $W_2 = 1$, iii) dim $W_1 = 2$.

Problem : Quickies
1. A square matrix Q is orthogonal if $Q^T Q = I$. Show that the determinant of any orthogonal matrix Q is always $+1$ or -1.
2. If the square matrix A is diagonalizable, so is any power of A^k.
3. If A is invertible and diagonalizable, then A^{-1} is also diagonalizable.
4. Are vectors $(1, 1, 1, 1)^T$, $(1, -1, -1, 1)^T$, $(1, 1, -1, -1)^T$ linearly independent?

5. Are the vectors $(1,1,1,1)^T$, $(1,-1,-1,1)^T$, $(1,1,-1,-1)^T$, $(1,2,1,-1)^T$, $(a,b,c,d)^T$ linearly independent?
6. If A and B are 3×3 matrices with $\det(A) = 3$, $\det(B) = 1$, find $\det((2A)^2(3B))$ and $\det(-AB^{-1})$!
7. Given a matrix with all entries equal to one, find one of its eigenvalues.
8. Can you find two matrices A and B so that $\text{rank}(A) = 5$, $\text{rank}(B) = 3$, and $\text{rank}(AB) = 4$?
9. What is the dimension of the subspace of $\mathbb{R}^4$ that consists of all vectors of the form (a, b, c, d) with $c = a - b$ and $d = a + b$.
10. Let A be a matrix and R be its reduced row echelon form. Is $\text{Row}(A) = \text{Row}(R)$? Is $\text{Col}(A) = \text{Col}(R)$? Is $\text{Null}(A) = \text{Null}(R)$?
11. How many vectors is there in any basis of $\mathbb{R}^6$?
12. Can an eigenvector of a matrix be associated with two distinct eigenvalues?
13. Is it true that every two matrices with the same spectrum have identical characteristic polynomials?
14. Given a 5×5 matrix A, what is the sum of algebraic multiplicities of eigenvalues of A? Can the sum of geometric multiplicities of eigenvalues of A be greater than 5?
15. Does the rotation matrix A_θ have any real eigenvalues for $0 < \theta < 180$?
16. Let A and B be diagonalizable $n \times n$ matrices. Is $A + B$ diagonalizable? How about A^{-1}?
17. If A is a 3×3 matrix with eigenvalues 1, 2, 3, what is $\text{trace}(A)$?