Moments and Centers of Mass

Note: In the following you may think of first moments as "balancing" moments. A body's first moments tell us about balance and about the torque the body exerts about different axes. You may think of second moments as "turning" moments which might be used to anaylze the energy in a rotating shaft.

Moments and Centers of Mass for thin sheets and plates (2 dimensional):

In the following, the function $\delta(x,y)$ gives the density of a thin sheet at the point (x,y).

Mass: $M = \int\int \delta(x,y)dA$

First Moments: $M_x = \int\int y\delta(x,y)dA, \quad M_y = \int\int x\delta(x,y)dA$

Coordinates of the Center of Mass: $\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}$

Centroid: Note that when δ is constant, the location of the center of mass becomes a feature of the object's shape and not of the material of which it is made. In such a case, the center of mass is often called the centroid of the shape. To find the centroid, set $\delta = 1$ and proceed to find $\bar{x}$ and $\bar{y}$ as above.

Moments of Inertia (second moments):

About the x-axis: $I_x = \int\int y^2\delta(x,y)dA$

About the y-axis: $I_y = \int\int x^2\delta(x,y)dA$

About the origin: $I_0 = \int\int (x^2+y^2)\delta(x,y)dA = I_x + I_y$

About the line L: $I_L = \int\int r^2(x,y)\delta(x,y)dA$

where $r(x,y)$ is the distance from the point (x,y) to L

Radii of gyration:

About the x-axis: $R_x = \sqrt{\frac{I_x}{M}}$

About the y-axis: $R_y = \sqrt{\frac{I_y}{M}}$

About the origin: $R_0 = \sqrt{\frac{I_0}{M}}$

Moments and Centers of Mass a solid (3 dimensional) :

In the following, the function $\delta(x,y,z)$ gives the density at the point (x,y,z).

Mass: $M = \int\int\int_E \delta(x,y,z)dV$

First Moments about the coordinate planes:

$$M_{yz} = \int\int\int_E x\delta(x,y,z)dV, \quad M_{xz} = \int\int\int_E y\delta(x,y,z)dV, \quad M_{xy} = \int\int\int_E z\delta(x,y,z)dV$$

Coordinates of the Center of Mass: $\bar{x} = \dfrac{M_{yz}}{M}, \quad \bar{y} = \dfrac{M_{xz}}{M}, \quad \bar{z} = \dfrac{M_{xy}}{M}$

Moments of Inertia (second moments):

About the x-axis: $I_x = \int\int\int_E (y^2+z^2)\delta(x,y,z)dV$

About the y-axis: $I_y = \int\int\int_E (x^2+z^2)\delta(x,y,z)dV$

About the z-axis: $I_z = \int\int\int_E (x^2+y^2)\delta(x,y,z)dV$

About the line L: $I_L = \int\int\int_E r^2(x,y,z)\delta(x,y,z)dV$

where $r(x,y.z)$ is the distance from the point (x,y,z) to L

Radii of gyration about the line L: $R_L = \sqrt{\dfrac{I_L}{M}}$

Moments and Centers of Mass for coil springs, thin rods and wires :

In the following, the function $\delta(x,y,z)$ gives the density at the point (x,y,z).

Mass: $M = \int_C \delta(x,y,z)ds$

First Moments about the coordinate planes:

$$M_{yz} = \int_C x\delta(x,y,z)ds, \quad M_{xz} = \int_C y\delta(x,y,z)ds, \quad M_{xy} = \int_C z\delta(x,y,z)ds$$

Coordinates of the Center of Mass: (3 dimensional) $\bar{x} = \dfrac{M_{yz}}{M}, \quad \bar{y} = \dfrac{M_{xz}}{M} \quad \bar{z} = \dfrac{M_{xy}}{M}$

Moments of Inertia (second moments):

About the x-axis: $I_x = \int_C (y^2+z^2)\delta(x,y,z)ds$

About the y-axis: $I_y = \int_C (x^2+z^2)\delta(x,y,z)ds$

About the z-axis: $I_z = \int_C (x^2+y^2)\delta(x,y,z)ds$

About the line L: $I_L = \int_C r^2(x,y,z)\delta(x,y,z)ds$

where $r(x,y,z)$ is the distance from the point (x,y,z) to L