

Integrability Properties of Nonlinearly Perturbed Euler Equations

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Abstract

We study the integrability properties of nonlinearly perturbed Euler equations (linear ordinary differential equations with one regular singular point in the complex plane plus a nonlinear perturbation) near the singular point. We allow for first integrals with essential singularities and give sufficient conditions for the nonintegrability of the equations in the complex domain. We extend normal form theorems for singular equations and argue that equivalence to normal forms captures the spirit of the poly-Painlevé test and is a powerful tool for a rigorous approach to nonintegrability.

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1 Introduction

1.1 The Poly-Painlevé Test

The poly-Painlevé test for integrability that was proposed by Martin Kruskal [1],[2] is a generalization of the Painlevé test, in the sense that movable branch points for the solutions are allowed, as long as the branching of the solutions is not “too rich”.

More precisely, consider a first order ODE in the complex plane

$$\frac{du}{dx} = f(u, x)$$

where f is analytic in a domain $D_u \times D_x \subset \mathbf{C}^2$. The solutions $\mathbf{u}(x)$ might be multi-valued, and yet there might exist an analytic first integral in $D_u \times D_x$: a non-constant function $F(u, x)$ such that $F(u(x), x)$ is constant for all the branches u of \mathbf{u} in D_x .

Let us fix some solution \mathbf{u} and a point $x \in D_x$ and consider the set

$$S_{\mathbf{u},x} = \{u(x) ; u \in \mathbf{u}\}$$

If, on the other hand, the set $S_{\mathbf{u},x}$ is dense in some open set $\tilde{D} \subset D_u$ (in which case we say that the solution \mathbf{u} has bad branching) then any continuous function F which is constant on the trajectories must be constant on \tilde{D} , and the equation has no continuous first integrals on $\tilde{D} \times D_x$.

This idea is illustrated in [2], [3] on Abel’s equation $\frac{du}{dx} = u^3 + x$, which is shown to be nonintegrable. The calculations use repeated integration on Riemann surfaces and are quite involved.

1.2 Motivation and Other Results in the Literature

Perhaps the simplest example of deducing nonintegrability (i.e. non-existence of holomorphic first integrals) using the poly-Painlevé test is provided by the following equation

$$xu' = \mu u + h(x, u) \tag{1}$$

where $h(x, u)$ is holomorphic at $x = u = 0$, having a zero of order 2 at $u = 0$ (i.e. $h(x, u) = u^2 h_1(x, u)$ with h_1 holomorphic at $(x, u) = (0, 0)$) and μ is a complex parameter.

In order to study the multi-valuedness of the solutions, the poly-Painlevé test designed by Martin Kruskal uses of a technique close to the α -method:

introducing a small parameter in the equation, calculating the series solutions and studying the multi-valuedness of its terms. This amounts to the study of the equation in a certain region of the phase space.

In our example (1), we choose to study the equation for small values of u . So we set up the equation in a regularly perturbed form by introducing a small parameter ϵ via the substitution $u = \epsilon U$. The equation becomes

$$xU' = \mu U + O(\epsilon) \tag{2}$$

and the solutions U can be found as perturbation series in ϵ :

$$U = U_0 + \epsilon U_1 + \epsilon^2 U_2 + \dots \tag{3}$$

The nonintegrability test is very easy to perform in this example because of the following features: the reduced equation (i.e. the equation with $\epsilon = 0$) has only one singular point ($x = 0$); the multi-valuedness of U_0 can be readily found and the nonintegrability (or integrability, depending on the value of μ) can be easily established.

Indeed, the test proceeds as follows.

First, the perturbation series for the solutions U is calculated. We solve (2) with any initial condition $U = k$ at $x = x_0$. The point x_0 is arbitrary (except 0 and ∞). Say, $x_0 = 1$. In terms of the initial dependent variable u , it means that we are considering the one-parameter family of solutions with initial condition $u(1) = \epsilon k$. But (if $k \neq 0$), the constant k can be absorbed in ϵ , hence we may take $k = 1$.

(At this point we should discuss the significance of the “artificially” introduced parameter ϵ : in fact, in our example ϵ is an initial condition and the expansion (3) is an expansion in the initial data.)

Martin Kruskal observed that the parameter ϵ which is introduced for the poly-Painlevé test is internal, meaning that the equations (2) corresponding to different values of $\epsilon \neq 0$ are equivalent. *However*, the *reduced* equation is not equivalent to the others; it is normally chosen to be “simpler” than the original, at least with respect to finding the multi-valuedness of the solutions.)

The reduced equation has the solution $U_0 = x^\mu$ hence $U \equiv U(x; \epsilon) = x^\mu + O(\epsilon)$. The solutions are multivalued (if μ is not an integer). The power series in ϵ converges for $|\epsilon|$ small enough, and x belonging to a closed path γ starting and ending at 1, encircling the origin (as will be shown in Lemma 1).

Assume there is a single-valued function F constant on the solutions of (1): $F(x, \epsilon U(x; \epsilon)) = \text{const}$. Hence, if $U^{ac}(x; \epsilon)$ denotes some analytic

continuation of $U(x; \epsilon)$ (on a closed path) we must have

$$F(x, \epsilon U(x; \epsilon)) = F(x, \epsilon U^{ac}(x; \epsilon)) \quad (4)$$

or, using the (convergent) ϵ -expansion,

$$F(x, \epsilon U_0(x) + \epsilon^2 U_1(x) + \dots) = F(x, \epsilon U_0^{ac}(x) + \epsilon^2 U_1^{ac}(x) + \dots) \quad (5)$$

Suppose we keep only the first term in the series above and discard the rest; we get

$$F(x, \epsilon U_0(x)) = F(x, \epsilon U_0^{ac}(x)) \quad (6)$$

or, more precisely,

$$F(x, \epsilon x^\mu) = F(x, \epsilon x^\mu e^{2n\pi i \mu})$$

for all integers n .

If μ is an irrational number, the set $\{e^{2n\pi i \mu}; n \in \mathbf{Z}\}$ is dense in the unit circle. Therefore, F cannot depend on the second variable. But then F must be constant, so the reduced equation has no single-valued first integrals. When the poly-Painlevé test is done in practice, one immediately concludes that the original equation (2) has no first integrals as well. This last step is intuitively appealing, but the question is: is it always correct?

To formulate more precisely the question, let us consider (4) for a fixed value of x , say $x = 1$ (for other values of x we may simply rescale ϵ) and denote $\phi(\epsilon) = \epsilon U^{ac}(1; \epsilon)$ where the analytic continuation is considered along a closed path encircling the origin once, counterclockwise. So ϕ is holomorphic in a neighborhood of the origin and $\phi(\epsilon) = \epsilon e^{2\pi i \mu} + O(\epsilon^2)$. With this notation equality (4) can be written as

$$F(1, \epsilon) = F(1, \phi(\epsilon))$$

which can be iterated and gives

$$F(1, \epsilon) = F(1, \phi(\epsilon)) = F(1, (\phi \circ \phi)(\epsilon)) = \dots = F(1, \phi^{\circ n}(\epsilon))$$

where

$$\phi^{\circ n}(\epsilon) \equiv (\phi \circ \dots \circ \phi)(\epsilon) = \epsilon e^{2n\pi i \mu} + O(\epsilon^2)$$

Therefore, $F(1, \cdot)$ must have the same values on all the iterates $\phi^{\circ n}(\epsilon)$ of $\phi(\epsilon)$. While the iterates $\epsilon e^{2n\pi i \mu}$ of the linear part of ϕ have values dense on a closed curve (a circle) it is not clear that the same will be true for the iterates of $\phi(\epsilon)$. We can not directly infer nonintegrability of the original equation based on the same property of its linear approximation, and the α -method needs more justification.

There are, in the literature, several (less general than the poly-Painlevé test) approaches to the study of nonintegrability in the complex plane.

One of the earliest relies on the α -method: a small parameter α is introduced in the equation (internal parameter), or is identified among the parameters of the equation (external parameter); it is assumed that there are *first integrals expressed as power series in α* (the Poincaré series). The terms of the series are calculated order by order. If one of them is not single-valued, then the sum of the series cannot be single-valued, hence non-integrability follows.

A more modern approach is Ziglin's theorem [4]. The technique consists of linearizing a system of differential equations around particular solutions (forming linear manifolds). The linearized equation is then studied. If its monodromy group is too large (i.e. the branches of solutions have too many values) then the linearized equation has no meromorphic first integrals. It is then shown that this implies that the original system has no *meromorphic* first integrals.

Ziglin's approach appears to be a particular instance of the poly-Painlevé test. Indeed, on one hand, Ziglin's theorem applies only to Hamiltonian systems. On the other hand, it can be viewed as a zero order poly-Painlevé test: we may introduce a small parameter in the equations in such a way, that the reduced equation is the linearized part (see [5] for an example). The linearized system is then studied: it may or may not turn out to be integrable. In the latter case, both tests yield nonintegrability. In the first case, Ziglin's theorem can infer nothing about the integrability properties of the original system, while in the poly-Painlevé test one would proceed to the study of the higher order terms of the solutions, which might display bad branching (hence, nonintegrability may be found).

Other variations on Ziglin's approach are based on the assumption that the first integrals must be regular near the linear manifold (see for example [6]) or algebraic (as in some results of Yoshida—see [3] and the references therein).

A common feature of all these nonintegrability tests is the presence of supplementary assumptions (playing an important role in the proofs) on the first integrals (regularity, meromorphicity or algebraic character). But these are somewhat artificial: in fact, a first integral with essential singularities (like, say the integral $F(x, u) = x \exp(u^{-1})$ for the equation $du/dx = u^2/x$) is, for most purposes, just as good as a meromorphic one—it implies that the trajectories are confined to a lower dimensional manifold.

On the other hand, in many cases meromorphicity is a quite strong condition: it implies that the reduced equation must have a *rational* (in u or/and

in x) first integral; analyticity, as an assumption, implies the existence of a polynomial integral. To illustrate this, consider the example of equation (1). Suppose that there is a first integral F which is regular at $u = 0$. We may then expand in power series:

$$\begin{aligned} F(x, u(x)) &= F(x, \epsilon U_0(x) + \epsilon^2 U_1(x) + \dots) = \\ &F(x, 0) + F_u(x, 0) (\epsilon U_0(x) + \epsilon^2 U_1(x) + \dots) + \\ &\frac{1}{2} F_{uu}(x, 0) (\epsilon U_0(x) + \epsilon^2 U_1(x) + \dots)^2 + \dots = \\ &F(x, 0) + \epsilon F_u(x, 0) U_0(x) + \epsilon^2 \left(F_u(x, 0) U_1(x) + \frac{1}{2} F_{uu}(x, 0) U_0(x)^2 \right) + \dots \end{aligned}$$

and if F is constant on the solutions, it follows that $F(x, 0) \equiv \text{const}$ and that $\partial_u^k F(x, 0) U_0^k$ is a first integral for the reduced equation (where k is the smallest number such that $\partial_u^k F(x, 0) \neq 0$). Similarly, if we only require meromorphicity of F , we get that a first integral of the reduced equation must have the form $f(x) U_0^k$ where k is an integer.

We thus find that for the equation (1) to have a first integral which is meromorphic at $u = 0$ it is necessary that μ be rational.

The same holds, of course, for the equation $xu' = \mu u$. However, this equation is in fact integrable for almost all the values of μ , namely for any μ not real: there is a single-valued first integral which has essential singularity at $u = 0$. Indeed, $ux^{-\mu}$ is constant on the solutions, and so is $\ln u - \mu \ln x$. The multi-valuedness of the last expression is $\ln u - \mu \ln x + 2n\pi i - 2m\mu\pi i$ where n, m are arbitrary integers. If μ is not real, let \mathcal{F} be a doubly periodic function, with periods $2\pi i$ and $2\mu\pi i$. Then the function $F(x, u) = \mathcal{F}(\ln u - \mu \ln x)$ is a *single-valued first integral* of the equation. Clearly, $F(x, \cdot)$ has essential singularity at $u = 0$ (accumulation of poles), for all x for which the function is defined ⁽¹⁾.

⁽¹⁾ There is more than one definition for an “essential singularity”

for functions of *several* complex variables. We will not deal with the subtle aspects of this issue, but merely prove “non-meromorphicity”.

Setting $h(x, u) = u^2$ in equation (1) (with μ not real) we get another example of integrable equations with singular first integrals. Indeed, the general solution is $u = -\mu/(1 + cx^{-\mu})$ (and there is one particular solution $u = 0$). A single-valued first integral can be obtained as in the case $h = 0$. Namely, from $c = -(\mu u^{-1} + 1)x^\mu$ we find that the function $F(x, u) = \mathcal{F}(\ln(\mu u^{-1} + 1) + \mu \ln x)$ is a single valued first integral if \mathcal{F} is a doubly periodic function, of periods $2\pi i$ and $2\pi i\mu$. As before, $F(x, \cdot)$ has essential singularity at $u = 0$.

We have thus found in the first order, non-linearly perturbed Euler equations (1) a class of examples on which the first integrals must be thought of as typically not meromorphic, and not algebraic. The question is then to find a rigorous method, or argument, for proving nonintegrability (if that is the case). The present paper aims at answering this question on a (more general) class of equations.

1.3 Description of the Main Results

The present work is a study of the integrability properties, in the complex domain, of a class of differential equations: the nonlinearly perturbed Euler equation

$$x^n u^{(n)} + c_{n-1} x^{n-1} u^{n-1} + \dots + c_0 u = h(x, u, u', \dots, u^{(n-1)}) \quad (7)$$

where $c_j \in \mathbf{C}$, h is a holomorphic function of $n + 1$ variables in a neighborhood of the origin in \mathbf{C}^{n+1} and has a zero of order 2 at $(u, u', \dots, u^{(n-1)}) = (0, 0, \dots, 0)$, and more generally, for systems

$$x u'(x) = M u(x) + a(x, u) \quad u(x), a(x, u) \in \mathbf{C}^n, x \in \mathbf{C} \quad (8)$$

where $a = (a_1, \dots, a_n)$, with $a_j(x, u)$ holomorphic for x in an annulus centered at 0, and u small, and having a zero of order 2 at $u = 0$; M is a constant matrix. (The substitution $(x \frac{d}{dx})^j u = u_{j+1}$, $j = 0, \dots, n - 1$ transforms the equation (7) into a system of the type (8) where $a_n = h(x, u_1, u_2/x, \dots)$ so a might be singular at $x = 0$.)

In order to perform the poly-Painlevé test in the region where the vector u is small, and x is of order 1, one can proceed as in the one-dimensional case: the substitution $u = \epsilon U$ in (8) yields

$$x U'(x) = M U(x) + \epsilon^{-1} a(x, \epsilon U) \quad , \quad \text{where } \epsilon^{-1} a(x, \epsilon U) = O(\epsilon) \quad (\epsilon \rightarrow 0) \quad (9)$$

The reduced ($\epsilon = 0$) equation (which is also the linearized equation) is studied in Section 2.1. We considered the question of existence of first integrals which are holomorphic functions of $n + 1$ variables on a domain $x \in D_x \subset \mathbf{C}$, $u \in D_u \subset \mathbf{C}^n$, where D_x contains a loop surrounding the point $x = 0$ (i.e. sufficiently general to permit analytic continuation of solutions around the branch point). Sufficient conditions for nonintegrability (within this class) are found. Also, when first integrals do exist, they are shown not to be meromorphic near the variety $u = 0$ (in generic cases).

The main issue is to prove that nonintegrability of the reduced equation implies nonintegrability of the original equation.

We need to specify the class of regularity for the first integrals considered for equations (8). We are not assuming that the integrals are meromorphic near $u = 0$ (and thus, do not necessarily have power series in ϵ). Instead, we deal with a more general class of functions:

Requirement R: We require that the integrals $F = F(x, u)$ be holomorphic on a domain of the form $D_x \times D_u$ where $0 \in \partial D_u$ and $D_x \subset \mathbf{C}$ is such that in any neighborhood of $x = 0$ there is a closed loop around the origin, contained in D_x . (For example, functions F holomorphic on an open set of the form $\mathbf{C}^{n+1} \setminus V$, where V is a holomorphic variety consisting of singularities of F satisfy the requirement.)

In Section 2.2 it is shown that, for almost all matrices M , equation (8) is holomorphically equivalent to its linear part (10), on a certain domain in the phase space. This type of result is usually referred to, in the literature, as “equivalence to a normal form” [7].

There are exceptional cases when the result of Section 2.2 does not hold and the following subsections are devoted to the study of the integrability properties in such cases. To this end, the multi-valuedness of solutions is studied.

In order to find the values of solutions on closed paths surrounding the origin, we first show, in section 2.3, that the solutions are defined on such paths. Then, in section 2.4, the form of the solutions after analytic continuation is given.

Section 2.6 gives sufficient conditions for the multi-valuedness of the solutions of (9) to form a set holomorphically equivalent to the one corresponding to the reduced equation (10). As a consequence, whenever the reduced equation is not integrable (and the sufficiency conditions fulfilled) in the class of first integrals specified above, so is the original equation, in the class of first integrals satisfying Requirement R.

There are still cases when the results of neither Section 2.2 nor Section 2.6 do apply. Section 2.6.3 deals with them in the one-dimensional case. Further investigation is needed for higher dimensions.

2 Main Results

Consider the equation (8). We will assume in what follows that the characteristic exponents of its linear part, i.e. the eigenvalues μ_1, \dots, μ_n of M , are distinct.

We may therefore assume that the matrix M is diagonal:
 $M = \text{diag}\{\mu_1, \dots, \mu_n\}$.

2.1 Integrability Properties of the Reduced Equation

We will first study the reduced equation

$$xU'(x) = MU(x) \tag{10}$$

We will distinguish several cases.

Remark 1 *If at least one of the characteristic exponents μ_j is not real, then the equation (10) has n independent single-valued first integrals.*

Proof

A fundamental system of solutions is $u_j = c_j x^{\mu_j}$, $j = 1, \dots, n$. Hence $c_j = u_j x^{-\mu_j}$, $j = 1, \dots, n$ are (typically multi-valued) first integrals. We will use them to produce single-valued ones.

Suppose that μ_1 is not real. Then the following functions are independent, single-valued first integrals: for each $j = 1, \dots, n$, if μ_j is not real, consider the function $F_j(x, u) = \mathcal{F}_j(\ln u_j - \mu_j \ln x)$ (where \mathcal{F}_j is a doubly periodic function, of periods $2\pi i$ and $2\pi i \mu_j$), and if μ_j is real, the function $F_j(x, u) = \mathcal{F}_j(\mu_1 \ln u_j - \mu_j \ln u_1)$ (where \mathcal{F}_j is a doubly periodic function, of periods $2\pi i \mu_1$ and $2\pi i \mu_j$). ■

The integrals that we found are, clearly, not meromorphic (near $u = 0$). The next proposition shows that this is the case for any single-valued first integrals.

Proposition 1 *Consider the equation (10). Assume that the numbers $1, \mu_1, \dots, \mu_n$ are linearly independent over \mathbf{Z} .*

Denote by μ_1, \dots, μ_q the real characteristic exponents, and μ_{q+1}, \dots, μ_n the non-real ones.

If the numbers $\Im \mu_{q+1}, \dots, \Im \mu_n$ are linearly independent over \mathbf{Z} , then there is no single-valued first integral meromorphic on a domain $D = D_x \times D_u$ with $0 \in D_u$.

Proof

Step I: we first show (as we stated in the introduction) that the existence of a meromorphic local first integral implies the existence of a rational homogeneous one.

Suppose that the equation (10) admits a single-valued first integral $F(x, u)$, meromorphic on D . Then F is a function of the n constants of integration $c_k = u_k x^{-\mu_k}$:

$$F(x, u) = \Phi(u_1 x^{-\mu_1}, \dots, u_n x^{-\mu_n}) \tag{11}$$

Fix $x \neq 0$ for which the function $F(x, \cdot)$ is defined (as a meromorphic function on D_u). Then (11) clearly implies that Φ is meromorphic on a neighborhood B of the origin.

Denote

$$\lambda_j = e^{2\pi i \mu_j} \quad (12)$$

Since F is single-valued, Φ must satisfy

$$\Phi(c_1, \dots, c_n) = \Phi(\lambda_1^{-1} c_1, \dots, \lambda_n^{-1} c_n) \quad (13)$$

for all c in a suitable neighborhood of the origin.

We introduce a small parameter α in the problem by the substitution $U = \alpha V$. Then V satisfies the same differential equation as U and there is a meromorphic first integral, namely $F(x, \alpha V) = \Phi(\alpha C_1, \dots, \alpha C_n)$ where $C_k = V_k x^{-\mu_k}$. Since Φ is meromorphic, there are two functions f, g , analytic near 0, such that $\Phi = f/g$. By expanding in Taylor series and collecting the terms in α we get

$$\Phi(\alpha C_1, \dots, \alpha C_n) = \frac{f(\alpha C_1, \dots, \alpha C_n)}{g(\alpha C_1, \dots, \alpha C_n)} = \frac{\sum_{r \geq r_0} \alpha^r P_r(C_1, \dots, C_n)}{\sum_{s \geq s_0} \alpha^s Q_s(C_1, \dots, C_n)}$$

where P_r, Q_s are polynomials (homogeneous of degrees r and s respectively) and by assumption $P_{r_0}, Q_{s_0} \neq 0$. By expanding further as a (convergent) power series in α , we get

$$\Phi(\alpha C_1, \dots, \alpha C_n) = \alpha^{r_0 - s_0} \left(\frac{P_{r_0}(C_1, \dots, C_n)}{Q_{s_0}(C_1, \dots, C_n)} + O(\alpha) \right) \quad (\alpha \rightarrow 0) \quad (14)$$

Introducing (14) in the equation $\Phi = \text{const}$ it is easy to see that P_{r_0}/Q_{s_0} is a first integral.

So we may assume that the function Φ is rational.

Step II: we show that the relation (13) implies that Φ does not depend on c_{q+1}, \dots, c_n .

We may assume that $|\lambda_j| \geq 1$ for all j . Indeed, we can substitute $c_j = d_j$ if $|\lambda_j| \geq 1$ and $c_j = d_j^{-1}$ if $|\lambda_j| < 1$ and the function Φ , in the new variables (d_1, \dots, d_n) , is also rational.

Since the numbers $1, \mu_1, \dots, \mu_q$ are real, and non-resonant, there exists a sequence $\{n_s\}_{s \in \mathbb{N}}$ of natural numbers such that (cf. (12))

$$\lim_{s \rightarrow \infty} \lambda_j^{-n_s} = 1 \quad , \quad j = 1, \dots, q \quad (15)$$

Iterating (13) we get

$$\Phi(c_1, \dots, c_n) = \Phi(\lambda_1^{-n_s} c_1, \dots, \lambda_n^{-n_s} c_n) \quad , \quad \text{for all } s \quad (16)$$

Denote $\Phi = P/Q$, where P, Q are polynomials.

Also denote by $c = (c', c'')$ the splitting of the vector $c = (c_1, \dots, c_n)$ with $c' = (c_1, \dots, c_q)$ and $c'' = (c_{q+1}, \dots, c_n)$.

Several cases are possible.

Case (i): $Q(c', 0) \neq 0$. Then, by taking the limit $s \rightarrow \infty$ in (16) we get (by (15) and since $|\lambda_j| > 1$ for $j = q + 1, \dots, n$)

$$\Phi(c', c'') = \Phi(c', 0)$$

so Φ does not depend on c'' .

Case (ii): $Q(c', 0) = 0, P(c', 0) \neq 0$. In this case replacing Φ by Φ^{-1} we get the same conclusion as above.

Case (iii): $Q(c', 0) = P(c', 0) = 0$. In order to find the limit in relation (16) we write $P(c', c'') = \sum_K P_K(c')(c'')^K$ (where $P_K \neq 0$ are polynomials and K are multi-indices) and similarly for Q . Then

$$\Phi(\lambda_1^{-n_s} c_1, \dots, \lambda_n^{-n_s} c_n) = \frac{\sum_K P_K(\lambda_1^{-n_s} c_1, \dots, \lambda_q^{-n_s} c_q) \left((\lambda'')^K \right)^{-n_s} (c'')^K}{\sum_L Q_L(\lambda_1^{-n_s} c_1, \dots, \lambda_q^{-n_s} c_q) \left((\lambda'')^L \right)^{-n_s} (c'')^L} \quad (17)$$

Note that because the numbers $\mathfrak{S}\mu_{q+1}, \dots, \mathfrak{S}\mu_n$ are linearly independent over \mathbf{Z} we have

$$|(\lambda'')^{K_1}| \neq |(\lambda'')^{K_2}| \quad \text{if } K_1 \neq K_2 \quad (18)$$

We order the real numbers $N_K = |(\lambda'')^K|, D_L = |(\lambda'')^L|$ which appear at the numerator, respectively denominator, in (17) with non-zero coefficients.

By (18), there are only three possible cases:

Case (a): there is only one largest term, namely $D_{L_0} = |(\lambda'')^{L_0}|$. Then taking the limit $s \rightarrow \infty$ in (16) we get

$$\Phi(c', c'') = 0$$

Case (b): there is only one largest term, namely $N_{K_0} = |(\lambda'')^{K_0}|$. Replacing Φ by Φ^{-1} , the same argument as above shows that the assumed first integral must in fact be a constant.

Case (c): the largest ones are $N_{K_0} = |(\lambda'')^{K_0}|$ and $D_{K_0} = |(\lambda'')^{K_0}|$. Then from (16) we get, in the limit $s \rightarrow \infty$,

$$\Phi(c_1, \dots, c_n) = \frac{P_{K_0}(c')(c'')^{K_0}}{Q_{K_0}(c')(c'')^{K_0}}$$

hence Φ does not depend on c'' .

Step III: we show that Φ is actually constant.

We use the fact that the set $\{(\lambda_1^n, \dots, \lambda_q^n) ; n \in \mathbf{Z}\}$ is dense in the q -dimensional torus. Iterating (16) we get

$$\Phi(c_1, \dots, c_q) = \Phi(\alpha_1 c_1, \dots, \alpha_q c_q) \text{ for all } \alpha_j \text{ with } |\alpha_j| = 1$$

so Φ must be a constant.

■

Note Results similar to Proposition 1 above can be obtained even in some cases when the numbers $\Im\mu_{q+1}, \dots, \Im\mu_n$ are linearly dependent over \mathbf{Z} . Consider for example the 2-dimensional case with complex-conjugate characteristic exponents: $\mu_{1,2} = \alpha \pm i\beta$ ($\beta > 0$). Two constants of motion are $c_1 = u_1 x^{-\alpha-i\beta}$ and $c_2 = u_2 x^{-\alpha+i\beta}$. It is more convenient to consider the constants $d_1 = c_1 c_2 = u_1 u_2 x^{-2\alpha}$ and $d_2 = c_2 / c_1 = u_2 / u_1 x^{2i\beta}$. If α is rational, then there is a rational first integral (a power of d_1). If, on the other hand, α is not rational, then any rational first integral has the form $\Phi(d_1, d_2)$ and the same reasoning as in Proposition 1 shows that Φ must be constant.

Remark 2 Assume that all the exponents μ_1, \dots, μ_n are real. Let q be the number of irrational exponents linearly independent over \mathbf{Z} . Then on any domain $D = D_x \times D_u$ with D_x containing a closed loop surrounding the point $x = 0$ the equation (10) has exactly $n - q$ independent holomorphic first integrals.

Proof

Suppose that μ_1, \dots, μ_N are rational, $\mu_{N+1}, \dots, \mu_{N+q}$ are irrational, linearly independent over \mathbf{Z} , and $\mu_{N+q+1}, \dots, \mu_n$ are irrational, linearly dependent over \mathbf{Z} on the previous $N + q$ exponents.

Suppose $F(x, u_1, \dots, u_n)$ is a single-valued first integral. After analytic continuation along a closed path around the origin $x = 0$, the solutions $u_j = c_j x^{\mu_j}$ have the new values $c_j \lambda_j x^{\mu_j}$ (where $\lambda_j = e^{2\pi i \mu_j}$). Therefore we must have

$$F(x, u_1, \dots, u_n) = F(x, \lambda_1 u_1, \dots, \lambda_n u_n) = \dots = F(x, \lambda_1^p u_1, \dots, \lambda_n^p u_n)$$

for all p integer. Since $\mu_{N+1}, \dots, \mu_{N+q}$ are independent over the integers, it follows that the set $\{(\lambda_{N+1}^p, \dots, \lambda_{N+q}^p) ; p \in \mathbf{Z}\}$ is dense in the torus $(S^1)^q$, therefore F cannot depend on u_{N+1}, \dots, u_{N+q} . So there are at most $n - q$ independent first integrals.

But there are obviously $n - q$ of them: if $\mu_j = n_j/m_j$ with n_j, m_j integers, then the integrals $u_j^{m_j} x^{-n_j}$, $j = 1, \dots, N$ are single-valued and if $p_j \mu_j = p_1 \mu_1 + \dots + p_{N+q} \mu_{N+q}$, with p_k integers, then $u_j^{p_j} u_1^{-p_1} \dots u_{N+q}^{-p_{N+q}}$ is a single-valued first integral. ■

2.2 Normal Form for the Flow

We now turn to the basic question, namely the significance of the integrability properties of the reduced equation (10) for the original equation (8).

From an intuitive point of view, since the reduced equation is obtained as an asymptotic approximation of the original equation (in a certain region of the phase space) we might expect that the two equations have, qualitatively, the same behavior. Moreover, in the class of equations that we study, the reduced equation is also the linear part of the original equation. The natural question is then to see whether the two equations are equivalent. Theorem 1 addresses this question.

We do not expect the equivalence to hold in a region of the phase space containing arbitrarily small values of x (because $a(x, u)$ might be singular at $x = 0$, so nonlinear terms might not be negligible for x small).

We give the following definition, similar to the ones found in [7],[8]:

Definition 1 *Let $C > 0, \nu > 0$. We say that $\{\mu_1, \dots, \mu_n\}$ is a collection of type $(C, \nu)_1$ if for all multi-indices $K \in \mathbf{N}^n$ with $|K| \geq 2$, and all $l \in \mathbf{Z}$, $s \in \{1, \dots, n\}$ we have*

$$|K \cdot \mu + l - \mu_s| > C(|K| + |l|)^{-\nu}$$

(where $K \cdot \mu \equiv K_1 \mu_1 + \dots + K_n \mu_n$, $|K| = K_1 + \dots + K_n$).

The set of multi-plets (μ_1, \dots, μ_n) forming collections of type $(C, \nu)_1$ for some C, ν have full Lebesgue measure in \mathbf{C}^n [7].

Theorem 1 (Normal Form of Equations in an Annulus Surrounding a Singular Point)

Consider the equation (8) with $a(x, u)$ holomorphic for $|u| < r'$, $r'' < |x| < r'''$. Assume that the characteristic exponents $\{\mu_1, \dots, \mu_n\}$ form a collection of type $(C, \nu)_1$.

Then the equation is biholomorphically equivalent to its linear part in a region $|u| < \rho'$, $\rho'' < |x| < \rho'''$ of the phase space.

The theorem is in the same spirit as the results known in the literature as “reduction to normal forms” [8] (e.g. normal linear form, normal form of an equation with periodic coefficients).

The proofs of these results (as well as of analogous ones for maps, and also of KAM theorem) share common characteristics: the holomorphic sought-for change of coordinates can be calculated as a formal series, term by term, but its convergence might be very difficult to prove in some cases because of the presence of small denominators. Instead, a sequence of approximations based on a generalization of Newton’s method is a more efficient approach.

The proof of Theorem 1 resembles the proof of the analogous result for systems with periodic coefficients. We give it in the Appendix.

The upshot is that, for almost all matrices M , *the integrability of the original equation (8) is equivalent to the integrability of the reduced equation (10)*, in the domain of the (extended) phase space where the two equations are holomorphically equivalent.

In particular, the number of single-valued first integrals of (8) on the whole phase space (less the variety $u = 0$) is *at most equal* to the number of first integrals of its linear part (some integrals of (8) might have branch points outside the domain where the two equations are equivalent).

For example, if the characteristic exponents are linearly independent over \mathbf{Z} and form a collection of type $(C, \nu)_1$, then equation (8) has no first integrals satisfying *Requirement R* of Section 1.2.

Also, under the conditions of the theorem, if the equation (8) does have first integrals satisfying *Requirement R* (so necessarily the characteristic exponents are resonant, or one of them is not real), they will almost always (cf. Proposition 1) be non-meromorphic near $u = 0$.

A natural question that arises here is whether there are cases (when the $(C, \nu)_1$ condition is not fulfilled) when the system and its linear part are not equivalent. This may indeed happen in the case of normal linear form at a regular point ([7], Ch.4, 1.3). This fact strongly suggests that the same could be true in our case.

2.3 Analytic Continuation of Solutions

The next question is to find the integrability properties of (8) when the condition $(C, \nu)_1$ does not hold. To this end, we study the multi-valuedness of the solutions. Since they could be branched at the origin, we study their analytic continuations for paths, in the x -plane, encircling the point $x = 0$. But we first have to show that the solutions are defined on such paths. The following lemma shows that the solutions of the equation (8) can be

analytically continued on paths winding a finite number of times around the origin in the complex x -plane.

Lemma 1 *Consider the equation (8) with $a(x, u)$ holomorphic in a neighborhood of the set*

$$\mathcal{D}_{\rho_0} = \{(x, u) \in \mathbf{C} \times \mathbf{C}^n ; \rho'' \leq |x| \leq \rho''', |u| \leq \rho'_0\}$$

$(0 < \rho'' < \rho''')$.

Let \mathcal{R} be a relatively compact subdomain of the universal covering of $\mathbf{C} \setminus \{0\}$ (i.e. the Riemann surface of the logarithm) that lies above the annulus $\rho'' < |x| < \rho'''$ in \mathbf{C} .

Then the solution u of the equation corresponding to the initial condition $u(x_0) = u_0$, where (x_0, u_0) belongs to the set

$$\mathcal{D}_\rho = \{(x, u) \in \mathbf{C} \times \mathbf{C}^n ; \rho'' \leq |x| \leq \rho''', |u| \leq \rho'\}$$

is holomorphic on \mathcal{R} provided that the number ρ' is small enough.

For our purposes we only need to consider two-sheeted coverings \mathcal{R} .

Proof

We will rewrite (8) as an integral equation and use a fixed point theorem to show that the equation has a solution.

The linear part of the equation has the solutions $U_k(x) = c_k x^{\mu_k}$, $k = 1, \dots, n$ (c_k are arbitrary constants). In vector notation, $U(x) = U_0(x)c$, where $U_0 = \text{diag}(x^{\mu_1}, \dots, x^{\mu_n})$ and $c = (c_1, \dots, c_n)^T$ is the general solution of (10). Using the variation of constants (i.e. writing $u(x) = U_0(x)c(x)$), the solutions u of (8) will satisfy the system

$$u(x) = U_0(x)c + U_0(x) \int_{x_0}^x t^{-1} U_0(t)^{-1} a(t, u(t)) dt \equiv N(u)(x) \quad (19)$$

We will need the following estimate for the function $a = (a_1, \dots, a_n)^T$. We assumed that $a_k(x, u)$ have a zero of order 2 at $u = 0$, i.e. a_k can be written as

$$a_k(x, u) = \sum_{j,l=1}^n u_j u_l a_{kjl}(x, u) \quad (20)$$

with a_{kjl} holomorphic near \mathcal{D}_{ρ_0} . Let M be an upper bound for a_{kjl} ($k, j, l = 1, \dots, n$) on \mathcal{D}_{ρ_0} and M' be an upper bound for $\partial_{s+1} a_{kjl}$ ($s, k, j, l = 1, \dots, n$).

Let $\rho' \leq \rho'_0$. Consider the Banach space \mathcal{B} of the n -tuples of analytic functions $u = (u_1, \dots, u_n)$ on (the piece of a Riemann surface) \mathcal{R} , continuous on $\overline{\mathcal{R}}$, with the sup norm:

$$\|u\| = \max_k \sup_{x \in \overline{\mathcal{R}}} |u_k(x)|$$

Then $a, (a_{1jl}, \dots, a_{njl}) \equiv a_{jl} \in \mathcal{B}$.

Let $N = (N_1, \dots, N_n)^T$ be the operator defined by (19). Then

$$N_k(u)(x) = x^{\mu_k} c_k + x^{\mu_k} \int_{x_0}^x t^{-1-\mu_k} a_k(t, u(t)) dt \quad (21)$$

Clearly, if $u \in \mathcal{B}$ then also $N(u) \in \mathcal{B}$.

Let $B_{\rho'}$ be the ball of radius ρ' in the Banach space \mathcal{B} .

We first show that if ρ' is small enough, and for small initial conditions u_0 , we have $N(\overline{B_{\rho'}}) \subset \overline{B_{\rho'}}$.

Indeed, let $(u_1, \dots, u_n) \in \overline{B_{\rho'}}$. Let Γ be an upper bound for the functions $|x^{\mu_k}|, |x^{-1-\mu_k}|, k = 1, \dots, n$ on \mathcal{R} . Let L be an upper bound for the lengths of paths on \mathcal{R} (up to homotopic equivalence).

Then

$$|N_k(u)(x)| \leq \Gamma |c_k| + L \Gamma^2 \sup |a_k(x, u)| \stackrel{\text{cf. (20)}}{\leq} \Gamma |c_k| + L \Gamma^2 n^2 M \rho'^2$$

For c_k and ρ' small enough (depending only on ρ'', ρ''', n) we have $|N_k(u)(x)| \leq \rho'$.

The constants c_k are related to the initial conditions u_0 by a linear invertible transformation:

$$u_0 = U_0(x_0)C \quad (22)$$

We conclude that for u_0 and ρ' small, the operator N maps the ball $\overline{B_{\rho'}}$ into itself.

Also, N is a contraction. Indeed,

$$|N_k(v)(x) - N_k(w)(x)| \leq L \Gamma^2 M''$$

where

$$M'' = \max \left\{ |a_k(x, v(x)) - a_k(x, w(x))| ; x \in \overline{\mathcal{R}} \right\}$$

Using the Taylor expansion with integral remainder and the relation (20)

$$|a_k(x, v) - a_k(x, w)| \leq$$

$$\left| \int_0^1 \sum_{j=1}^n (v_j - w_j) \partial_{j+1} a_k(x, w_0, \dots, w_{j-1}, w_j + (v_j - w_j)t, v_{j+1}, \dots, v_n) dt \right| \leq$$

$$\max_{j=1, \dots, n} |v_j - w_j| \max_{j=2, \dots, n+1} \max_{(x, v) \in \mathcal{D}_\rho} |\partial_j a_k(x, v)| \leq$$

$$\max_j |v_j - w_j| (2nM|v| + n^2 M' |v|^2) \leq \max_j |v_j - w_j| (2nM\rho' + n^2 M'(\rho')^2)$$

Applying the above inequality to estimate $|a_k(x, v(x)) - a_k(x, w(x))|$ we get

$$|N_k(v)(x) - N_k(w)(x)| \leq L \Gamma^2 (2nM\rho' + n^2M'(\rho')^2) \max_j \max_{x \in \overline{\mathcal{R}}} |v_j(x) - w_j(x)|$$

which shows that N is a contraction for ρ' small.

Therefore the operator N has a fixed point in $\overline{B_\rho}$ and the lemma is proved. ■

2.4 The Monodromy Map

We now study the multi-valuedness of the solutions.

Let $\mathcal{R}, \mathcal{D}_\rho$ be as in Lemma 1 and let $(x_0, u_0) \in \mathcal{D}_\rho$. Denote by $u(x; x_0, u_0)$ the solution of the equation (8) with the initial condition u_0 at $x = x_0$. Consider a path γ in \mathcal{R} , starting and ending above x_0 , whose projection on the complex plane encircles the origin once, counterclockwise. After analytic continuation along γ , the solution has a new value, $u^+(x_0; x_0, u_0)$. The map Φ_{x_0} given by

$$\Phi_{x_0}(u_0) = u^+(x_0; x_0, u_0) \quad (23)$$

is the monodromy map at x_0 on the annulus covered by \mathcal{R} , and is defined for $u_0 \in \mathbf{C}^n$, $|u_0| < \rho'$.

Lemma 2 *Let $(x, u) \in \mathcal{D}_\rho$, $u = (u_1, \dots, u_n)$.*

The monodromy map has the form

$$\Phi_x(u_1, \dots, u_n) = (\Phi_{x,1}(u_1, \dots, u_n), \dots, \Phi_{x,n}(u_1, \dots, u_n))$$

where $\Phi_{x,j}$ are holomorphic in a neighborhood of $u = 0$ and

$$\Phi_{x,j}(u_1, \dots, u_n) = \lambda_j u_j + O(u_k u_l) \quad (u \rightarrow 0)$$

Proof

Note first that the solutions with initial conditions in \mathcal{D}_ρ depend holomorphically on the constants c_1, \dots, c_n (since they are holomorphic in the initial value of u (for u small), and the constants c_1, \dots, c_n are related to the initial conditions by a linear relation (22)).

From relations (19),(20) we see that

$$u_k(x) = c_k x^{\mu_k} + \tilde{u}_k(x) \quad , \quad k = 1, \dots, n \quad (24)$$

where $\tilde{u}_k(x)$ is holomorphic in c_1, \dots, c_n for c_k small and $x \in \mathcal{R}$, and $\tilde{u}_k(x) = O(c_j c_l)$ ($c \rightarrow 0$).

After analytic continuation on a closed path γ in the x -plane the u_k become

$$u_k^+(x) = c_k \lambda_k x^{\mu_k} + \tilde{u}_k^+(x) \quad , \quad k = 1, \dots, n \quad (25)$$

where $\tilde{u}_k^+(x) = O(c_j c_l)$ ($c \rightarrow 0$).

Using the holomorphic implicit function theorem in equations (24), (25), we can solve for c in terms of u for small enough c . We get

$$u_j^+ = u_j \lambda_j + \tilde{u}_k^+ \quad , \quad j = 1, \dots, n \quad (26)$$

where \tilde{u}_k^+ is holomorphic for small u and $\tilde{u}_k^+ = O(u_k u_l)$. ■

2.5 The Local Form of the First Integrals

Since the solutions u depend holomorphically on the constants c for small c and $x \neq 0$ fixed, and satisfy (24), we can solve for c in terms of u , for c and u small, and obtain $c_k = c_k(x, u)$, $k = 1, \dots, n$. These are in fact n independent first integrals. They are holomorphic in u and the coefficients of the power series in u are holomorphic functions of x on \mathcal{R} (so they may be branched at $x = 0$).

2.6 Normal Form for the Monodromy Map

2.6.1 Some Classic Results

We will first reproduce some basic facts following [8].

The numbers $\lambda_1, \dots, \lambda_n$ are called *non-resonant* if there are no integers k_1, \dots, k_n such that $k_l \geq 0$, $\sum k_l \geq 2$ and $\lambda_j = \lambda_1^{k_1} \dots \lambda_n^{k_n}$.

A collection of eigenvalues $(\lambda_1, \dots, \lambda_n)$ *belongs to the Poincaré domain* if the moduli of the eigenvalues are all smaller or all greater than 1.

The complement of the Poincaré domain is *the Siegel domain*.

Let C, ν be positive constants. A collection of eigenvalues $(\lambda_1, \dots, \lambda_n)$ is called *collections of multiplicative type* (C, ν) if

$$|\lambda_j - \lambda_1^{k_1} \dots \lambda_n^{k_n}| \geq C(k_1 + \dots + k_n)^{-\nu}$$

for all $j = 1, \dots, n$, $k_l \geq 0$, $\sum k_l \geq 2$.

The set of collections $(\lambda_1, \dots, \lambda_n)$ which are not of multiplicative type (C, ν) for any C has measure 0 if $\nu > (n - 1)/2$.

Poincaré's Theorem *If at a fixed point the collection of the eigenvalues of the linear part of a holomorphic diffeomorphism belongs to the Poincaré domain, and they are non-resonant, then in a neighborhood of the fixed point*

the mapping can be reduced to its linear part by means of a biholomorphic diffeomorphism.

Siegel's Theorem *If the collection of eigenvalues of the linear part of a holomorphic diffeomorphism at a fixed point has multiplicative type (C, ν) for some C, ν , then the diffeomorphism is biholomorphically equivalent to its linear part at the fixed point.*

2.6.2 Eigenvalues Belonging to the Poincaré Domain

Assume that at least one of the μ_k is not real. The reduced equation (10) is then integrable (cf. Remark 1). In some cases, this implies integrability of the original equation in a region of the phase space.

Proposition 2 *Consider the equation (8) with the characteristic exponents μ_1, \dots, μ_n , not all real.*

Assume that $\lambda_1, \dots, \lambda_n$ (where $\lambda_k = e^{2\pi i \mu_k}$) satisfy the hypothesis of Poincaré's theorem (i.e. $\Im \mu_k > 0, k = 1, \dots, n$ or $\Im \mu_k < 0, k = 1, \dots, n$).

Let $\mathcal{R}, \mathcal{D}_\rho$ be as in Lemma 1.

Then the equation (8) has n independent first integrals, holomorphic on the (interior of the) domain $\mathcal{D}_\rho \setminus V$, where V is the variety $u = 0$ in $\mathbf{C} \times \mathbf{C}^n$.

There are no first integrals meromorphic near V .

Note

The first integrals guaranteed by Proposition 2 above may not be single-valued outside the domain \mathcal{D}_ρ .

Proof

We study the multi-valuedness of the first integrals c_k and show that we can find uniformizing functions.

Let u be a solution starting at an initial point in the interior of \mathcal{D}_ρ . Then u has the form (24) for some constants c_k . After analytic continuation on a closed path in the x -plane the functions u_k have the form (25).

Since u^+ is again a solution of the equation, there are new constants c_k^+ such that

$$u_k^+(x) = c_k^+ x^{\mu_k} + \tilde{u}_k^+(x) \quad , \quad k = 1, \dots, n \quad (27)$$

where $\tilde{u}_k^+(x)$ depend holomorphically on c^+ (near the origin) and $\tilde{u}_k^+(x) = O(c_j^+ c_l^+) (c^+ \rightarrow 0)$.

(In fact, c_k^+ is the value of the first integral c_k after analytic continuation in x around the origin.)

We consider the equations (25), (27) and solve for c_k^+ , $k = 1, \dots, n$ in terms of c_k , $k = 1, \dots, n$: $c_k^+ = f_k(c_1, \dots, c_n)$, $k = 1, \dots, n$ where $f =$

(f_1, \dots, f_n) is a biholomorphism in a neighborhood of the origin and

$$f_k(c_1, \dots, c_n) = \lambda_k c_k + O(c_j c_l) \quad , \quad k = 1, \dots, n \quad (28)$$

(Note that the functions f_k do not depend on x because c_k and c_k^+ are independent of x .)

By the Poincaré theorem, there exists a biholomorphic change of coordinates near the origin, $c_k = \phi_k(d_1, \dots, d_n)$, $k = 1, \dots, n$ such that

$$f_k \circ (\phi_1, \dots, \phi_n)(d_1, \dots, d_n) = \phi_k(\lambda_1 d_1, \dots, \lambda_n d_n), \quad k = 1, \dots, n$$

We look for n independent first integrals; they are functions of the constants of integration $\Phi(c_1, \dots, c_n)$. Since we are looking for single-valued first integrals, we must have

$$\Phi(c_1, \dots, c_n) = \Phi(c_1^+, \dots, c_n^+)$$

or, equivalently,

$$\Phi(c_1, \dots, c_n) = \Phi(f_1(c_1, \dots, c_n), \dots, f_n(c_1, \dots, c_n))$$

so

$$\Phi \circ (\phi_1, \dots, \phi_n)(d_1, \dots, d_n) = \Phi \circ (\phi_1, \dots, \phi_n)(\lambda_1 d_1, \dots, \lambda_n d_n)$$

Therefore Φ_1, \dots, Φ_n are independent first integrals for (8) if and only if $\Phi_1 \circ (\phi_1, \dots, \phi_n), \dots, \Phi_n \circ (\phi_1, \dots, \phi_n)$ are independent first integrals for (10). Remark 1 and Proposition 1 conclude the proof. ■

2.6.3 Eigenvalues of Absolute Value 1

Assume now that all the μ_k are real.

As in Proposition 2, one may use Siegel's theorem (for each fixed x) to find sufficient conditions for the monodromy map to be holomorphically conjugate to its linear part (i.e. to the monodromy map of the linearized equation).

However, the condition that the characteristic exponents μ_1, \dots, μ_n form a collection of type $(C, \nu)_1$ for some $C, \nu > 0$ is equivalent to the condition that $\lambda_1, \dots, \lambda_n$ form a collection of multiplicative type (C, ν) for some $C, \nu > 0$ necessary for Siegel's Theorem.

Therefore the conditions for nonintegrability based on Siegel's theorem are equivalent to those assumed in Theorem 1.

Stronger results may be obtained if one could use (stronger) sufficient conditions for topological equivalence of maps. One such result is available in the case $n = 1$.

Proposition 3 *Consider the equation (1) with μ irrational.*

Let \mathcal{D}_ρ be such that Lemma 1 holds.

Then there are no first integrals, holomorphic on \mathcal{D}_ρ .

Proof

Suppose there exists F , single-valued first integral, holomorphic on \mathcal{D}_ρ . Fix $x \in \mathcal{R}$ such that $F(x, \cdot)$ is defined a.e. on $|u| < \rho'$.

Since F is single-valued, $F(x, u(x))$ must have the same value after the analytic continuation of the solution $u(x)$:

$$F(x, u) = F(x, \Phi_x(u)) \tag{29}$$

for all u in the domain of $F(x, \cdot)$.

Since the monodromy map is holomorphic in a neighborhood of the origin, and satisfies $\Phi_x(u) = \lambda u + O(u^2)$ with $\lambda = \exp(2\pi i \mu)$, μ irrational, by Denjoy's theorem [7] Φ_x is topologically conjugate to its linear part: there exists a local homeomorphism ϕ such that $\Phi_x(\phi(v)) = \phi(\lambda v)$.

Relation (29) becomes

$$F(x, \phi(v)) = F(x, \phi(\lambda v))$$

which can now be iterated

$$F(x, \phi(v)) = F(x, \phi(\lambda v)) = F(x, \phi(\lambda^2 v)) = \dots = F(x, \phi(\lambda^n v))$$

for all n integer.

Since the set $\{\lambda^n ; n \in \mathbf{Z}\}$ is dense on the unit circle, and F, ϕ are continuous, it follows that

$$F(x, \phi(v)) = F(x, \phi(\alpha v))$$

for all α on the unit circle. So $F(x, \cdot)$ has the same value on a closed curve in the complex u plane, hence cannot depend on u . It follows that F must be constant. ■

3 Conclusions

We considered the nonlinearly perturbed Euler equation (8) in the case when the characteristic exponents μ_1, \dots, μ_n of the linear part are distinct and non-resonant (i.e. if $k_1 \mu_1 + \dots + k_n \mu_n - \mu_j \notin \mathbf{Z}$ for any non-negative integers k_1, \dots, k_n with $k_1 + \dots + k_n \geq 2$ and $j \in 1, \dots, n$.)

If the numbers μ_1, \dots, μ_n form a collection of type $(C, \nu)_1$ then the equation is holomorphically equivalent to its linear part for x in an annulus and u small. (The set of multiplets (μ_1, \dots, μ_n) for which the $(C, \nu)_1$ -type condition fails for all $C, \nu > 0$ has measure zero in \mathbf{C}^n .)

As a consequence, the integrability properties of equation (8) and of its linear part are the same, on the above domain of the phase space, for almost all matrices M . Furthermore:

(i) if in addition, all the exponents are real, equation (8) has no single-valued first integrals.

(ii) if one of the exponents is not real, and a stronger non-resonance conditions fulfilled, then any single-valued first integral of (8) (if there is any) is not meromorphic near the variety $u = 0$.

If the condition $(C, \nu)_1$ is not fulfilled, but $\Im \mu_j > 0, j = 1, \dots, n$, or $\Im \mu_j < 0, j = 1, \dots, n$ then the monodromy map is holomorphically conjugate with its linear part; therefore all first integrals are not meromorphic near $u = 0$.

The case $n = 1$ can be more completely analyzed: if the characteristic exponent is irrational, then the equation is nonintegrable.

In the resonant case, the equivalence of the equation with its linear part cannot be in general obtained, not even by formal power series changes of coordinates. Instead, some terms of the nonlinear part are asymptotically significant and should be kept when considering a normal form for the equation.

It would be interesting to see under which conditions topological equivalence of the monodromy map with its linear part implies equivalence of the integrability properties (since it apparently is a weaker condition than that of holomorphic equivalence). For the case of *regular* vector fields (i.e. equations of the form $v'(z) = Mv(z) + a(v, z)$, with a holomorphic at the origin) we note, however, the following statement in [7] (Ch.5, 6.1): “A vector field germ with non-resonant linear part in the complex plane is either analytically equivalent to its linear part [...], or is not even topologically equivalent to it (V.A. Nayshul).”

4 Appendix

4.1 Statement of the Result

We consider the system

$$zv'(z) = Mv(z) + a(v, z) \quad , \quad v \in \mathbf{C}^n \quad , \quad z \in \mathbf{C} \quad (30)$$

We assume that:

- (i) M is a constant $n \times n$ matrix, with the spectrum consisting of n distinct eigenvalues μ_1, \dots, μ_n ;
- (ii) $a = (a^1, \dots, a^n)$ with a^j holomorphic functions, defined on

$$\Pi_r = \{(v, z) \in \mathbf{C}^n \times \mathbf{C} ; |v| < r', r'' < |z| < r'''\} \quad (31)$$

(where $r \equiv (r', r'', r''')$, $r' > 0, 0 \leq r'' < r'''$ and $|v| \equiv \max\{|v_1|, \dots, |v_n|\}$) and a is continuous on the closure $\overline{\Pi_r}$ of the domain;

- (iii) a has a zero of order 2 at $v = 0$:

$$a^j(v, z) = \sum_{|K| \geq 2, l \in \mathbf{Z}} a_{K,l}^j v^K z^l, \quad j = 1, \dots, n$$

(where $K \equiv (K_1, \dots, K_n) \in \mathbf{N}^n$, $|K| \equiv K_1 + \dots + K_n$, $v^K \equiv v_1^{K_1} \dots v_n^{K_n}$);

- (iv) we may also assume that M is diagonal (after a linear substitution).

The main result is the following:

Theorem 2 *If the eigenvalues of M form a collection of type $(C, \nu)_1$ then the system (30) is holomorphically equivalent, in a domain $\Pi_\rho \subset \Pi_r$, to its linear part.*

The following subsections of the Appendix contain the proof of Theorem 2. It follows the main steps of the proofs in [8], Sec.12 (for the analytic reduction of analytic circle diffeomorphisms to rotations) and Sec.28 (for the local normal form of mappings at a fixed point).

4.2 The Homological Equation

Consider the following equation for h :

$$Mh(y, z) + a(y, z) - zh_z(y, z) - D_y h(y, z)M y = 0, \quad y, h(y, z) \in \mathbf{C}^n \quad (32)$$

where a, M satisfy the conditions (i),..., (iv).

The homological equation (32) has a formal power series solution

$$h^j(y, z) = \sum_{|K| \geq 2, l \in \mathbf{Z}} h_{K,l}^j y^K z^l \quad \text{where} \quad h_{K,l}^j = \frac{a_{K,l}^j}{K \cdot \mu + l - \mu_j} \quad (33)$$

Notation: $\Pi_{r,\delta} = \{(y, z) \in \mathbf{C}^n \times \mathbf{C} ; |y| < r'e^{-\delta}, r''e^\delta < |z| < r'''e^{-\delta}\} \subset \Pi_r$

Lemma 3 *The series (33) converges and there exists a constant $\alpha > 0$ (depending only on C, ν, n) such that*

$$\max_{\overline{\Pi_{r,\delta}}} |h^j(y, z)| \leq \delta^{-\alpha} \max_{\overline{\Pi_r}} |a^j(y, z)|$$

for every $\delta \in (0, 1/2)$ and every r ($r', r''' > 0, r'' \geq 0$).

Proof Using Cauchy estimates for a^j , and for $(y, z) \in \overline{\Pi_{r,\delta}}$, we have

$$h_+^j := \left| \sum_{|K| \geq 2, l \geq 0} h_{K,l}^j y^K z^l \right| \leq C^{-1} \max_{\overline{\Pi_r}} |a^j(y, z)| \sum_{|K| \geq 2, l \geq 0} (|K| + l)^\nu e^{-\delta(|K|+l)}$$

From

$$\begin{aligned} \mathcal{N}_{n,p} &:= \# \{(K_1, \dots, K_n, l) ; K_j \geq 0, l \geq 0, |K| + l = p\} = \\ &= \binom{n+p}{n} \leq c_n p^n \end{aligned}$$

and $p^{n+\nu} e^{-\delta p/2} \leq C_1 \delta^{-(n+\nu)}$ for $p \geq 2$ (where C_1 depends only on $n + \nu$, and not on δ) we have

$$h_+^j \leq 2C^{-1} c_n C_1 \delta^{-(n+1+\nu)} \max_{\overline{\Pi_r}} |a^j(y, z)|$$

Similar estimates hold for $h_-^j := \left| \sum_{|K| \geq 2, l < 0} h_{K,l}^j y^K z^l \right|$. \blacksquare

4.3 Norms

We will use the following norm: for functions $\phi = (\phi^1, \dots, \phi^n)$ holomorphic on Π_r and continuous on $\overline{\Pi_r}$, with $\phi(0, z) = 0$ define

$$\|\phi\|_r = \sup_{\overline{\Pi_r}} \frac{|\phi(y, z)|}{|y|}$$

We denote by $\|\cdot\|_{r,\delta}$ the corresponding norm on the domain $\Pi_{r,\delta}$.

Let f be a scalar function, regular as above: $f(y, z) = \sum_{|K| \geq 1, l \in \mathbf{Z}} f_{K,l} y^K z^l$

Remark 3

$$|f_{K,l}| \leq \begin{cases} \|f\|_r (r')^{-|K|+1} (r''')^{-l} & \text{for } l \geq 0 \\ \|f\|_r (r')^{-|K|+1} (r'')^{-l} & \text{for } l < 0 \end{cases}$$

Remark 4 *There exists a positive constant α , depending only on n , such that, for all $\delta \in (0, 1/2)$ we have $\max_{\overline{\Pi_{r,\delta}}} |f(y, z)| \leq r' \delta^{-\alpha} \|f\|_r$.*

The proof follows Remark 3 and estimates as in Lemma 3.

Remark 5

$$\|h^j\|_{r,\delta} \leq \delta^{-\alpha} \|a^j\|_r$$

Remark 6 *Let f be holomorphic on Π_r , continuous on $\overline{\Pi_r}$, with $f(0, z) = 0$. Then*

$$\max_{\overline{\Pi_{r,\delta}}} \left| \frac{\partial f}{\partial y_j} \right| \leq (1 - e^{-\delta})^{-1} \|f\|_r$$

The proof uses the Cauchy formula for $(y, z) \in \Pi_{r,\delta}$.

4.4 The Remainder

Given the equation (30) we change the dependent variable v to the new variable y using the substitution $v = y + h(y, z)$, where h is the solution of the homological equation (32). The equation for y is

$$zy'(z) = My(z) + \mathcal{R}(y, z) \tag{34}$$

$$\text{where } \mathcal{R}(y, z) = (I + D_y h)^{-1} [a(y + h(y, z), z) - a(y, z)]$$

Lemma 4 *Let α be as in Lemma 3 and Remarks 4,5.*

There exists a constant $\mathcal{K} \geq 2$ such that for all $\beta > 2\alpha + 2$, $\delta \in (0, \mathcal{K}^{-1})$ and r : if $\|a\|_r \leq \delta^\beta$, then the matrix $I + D_y h(y, z)$ is invertible for $(y, z) \in \Pi_{r,\delta}$ and the following estimates hold:

$$\max_{(y,z) \in \overline{\Pi_{r,\delta}}} \|D_y h(y, z)\| \leq C_4 \delta \quad \text{and} \quad \max_{(y,z) \in \overline{\Pi_{r,\delta}}} \|(I + D_y h(y, z))^{-1}\| \leq C_3$$

where the constants C_3, C_4 do not depend on r or δ .

Proof Using Remarks 6 and 5

$$\max_{\overline{\Pi_{r,\delta}}} \|D_y h(y, z)\| \leq (1 - e^{-\delta/2})^{-1} \sqrt{n} \delta^{\beta-\alpha} 2^\alpha \leq 2e^{1/4} \sqrt{n} \delta 2^\alpha \equiv C_4 \delta$$

For $\mathcal{K} \geq 2$ be large enough, and $\delta \in (0, \mathcal{K}^{-1})$ we have

$$\max_{\overline{\Pi_{r,\delta}}} \|D_y h(y, z)\| < C_4 \mathcal{K}^{-1} \equiv C'_3 < 1 \tag{35}$$

The lemma holds with $C_3 = (1 - C'_3)^{-1}$. ■

Lemma 5 Assume $\delta > 0$ is such that the map $(y, z) \rightarrow (y + h(y, z), z)$ is well defined on $\Pi_{r, \delta}$, and its image J is relatively compact in Π_r . Then $\|a(y + h(y, z), z) - a(y, z)\|_{r, \delta} \leq n(1 - e^{-\delta})^{-1} \delta^{-\alpha} \|a\|_r^2$

The proof uses Remarks 5 and 6.

Lemma 6 Let \mathcal{K} be the constant in Lemma 4 and $\beta > 2\alpha + 2$.

There exists a constant $\mathcal{K}_0 \geq \mathcal{K}$ with the following property: assuming that $\delta \in (0, \mathcal{K}_0^{-1})$ and r are such that the image of the domain $\Pi_{r, \delta}$ under the map $(y + h(y, z), z)$ is relatively compact in Π_r , we have: if $\|a\|_r < \delta^\beta$, then

$$\|\mathcal{R}\|_{r, \delta} \leq \delta^{-1-2\alpha} \|a\|_r^2$$

Proof Combining the results of Lemma 4 and Lemma 5

$$\|\mathcal{R}\|_{r, \delta} \leq C_3 n (1 - e^{-\delta})^{-1} \delta^{-\alpha} \|a\|_r^2 \equiv C_5 (1 - e^{-\delta})^{-1} \delta^{-\alpha} \|a\|_r^2 \leq \delta^{-1-2\alpha} \|a\|_r^2$$

where the last inequality holds for δ small enough. \blacksquare

4.5 The Iteration

We construct a sequence of holomorphic substitutions $v = y + h_k(y, z)$ which take equations of form (30) (with the nonlinear term a_k instead of a , defined on Π_{r_k}) into equations of the same form, but with the nonlinear terms a_{k+1} closer to 0 than a_k , and defined on smaller domains $\Pi_{r_{k+1}} \subset \Pi_{r_k}$.

We first choose the numbers $r_0 = (r'_0, r''_0, r'''_0), \delta_0, N$. Using them as starting points, we define inductively, for $k \geq 0$ the sequences:

$$\delta_{k+1} = \delta_k^{3/2}, \quad r'_{k+1} = r'_k e^{-\delta_k}, \quad r''_{k+1} = r''_k e^{\delta_k}, \quad r'''_{k+1} = r'''_k e^{-\delta_k}$$

We then construct a sequence of functions as follows. We start with the system (30) satisfying (i) to (iv). Suppose $\Pi_{r_0} \subset \Pi_r$. Let $a_0 = a$. We solve the homological equation (32) and we get the solution $h \equiv h_0$. After the substitution $v = y + h_0(y, z)$ the system for y is (34); it has the form (30) where the nonlinearity is now the remainder \mathcal{R} . Set $a_1 = \mathcal{R}$, defined on Π_{r_1} , and solve the homological equation (with a_1 in place of a) to get the solution h_1 , and so on.

Fix r_0 so that $\Pi_{r_0} \subset \Pi_r$. Let $0 < \delta_0 < \mathcal{K}_0^{-1}$. Then

$$r'_{k+1} = r'_0 e^{-(\delta_0 + \delta_1 + \dots + \delta_k)} = r'_0 e^{-(\delta_0 + \delta_0^{3/2} + \dots + \delta_0^{(3/2)^k})} \equiv r'_0 e^{-\Delta_k(\delta_0)}$$

and similarly for r''_{k+1}, r'''_{k+1} .

The sequence Δ_k of continuous functions on $[0, \mathcal{K}_0^{-1}]$ converges uniformly to a continuous function Δ . Clearly, $\Delta(0) = 0$ and for $\delta_0 > 0$, $\Delta(\delta_0) > 0$. We choose δ_0 small enough so that $\rho'' < \rho'''$, where

$$r_k'' < r_0'' e^{\Delta(\delta_0)} \equiv \rho'' \quad , \quad \rho''' \equiv r_0''' e^{-\Delta(\delta_0)} < r_k'''$$

We also choose δ_0 sufficiently small, so that the following condition holds: $e^{C_4 \Delta(\delta_0)} < \frac{3}{2}$, where C_4 is the constant of Lemma 4.

We impose a new restriction to r_0' (besides $r_0' \leq r$). The function $a(y, z)$ has a zero of order 2 at $y = 0$ and is continuous on $\overline{\Pi_r}$. Therefore, there exists a constant C_a such that $|a(y, z)| < C_a |y|^2$ on $\overline{\Pi_r}$. So $\|a\|_{r_0} < C_a r_0'$. Therefore, for r_0' small we will have $\|a\|_{r_0} \leq \delta_0^N$.

Denote $r_0' e^{-\Delta(\delta_0)} \equiv \rho'$. Fix $\beta > 2\alpha + 2$.

We finally choose $N \geq \beta$ so that if the image of $\Pi_{r_{k+1}}$ under the map

$$V_k(y, z) = (y + h_k(y, z), z) \tag{36}$$

is relatively compact in Π_{r_k} , and if $\|a_k\|_{r_k} < \delta_k^N$ then $\|a_{k+1}\|_{r_{k+1}} < \delta_{k+1}^N$: for $\|a_k\|_{r_k} < \delta_k^N \leq \delta_k^\beta$ we have, by Lemma 6,

$$\|a_{k+1}\|_{r_{k+1}} \leq \delta_k^{-1-2\alpha} \|a_k\|_{r_k}^2 \leq \delta_k^{2N-1-2\alpha} \leq \delta_{k+1}^N$$

where the last inequality holds provided that $3N/2 \leq 2N - 1 - 2\alpha$. So we must have $N \geq \max\{\beta, 2 + 4\alpha\}$.

4.6 Convergence of the Iteration

Remark 7 1) The image of $\Pi_{r_{k+1}}$ under the map V_k given by (36) is relatively compact in Π_{r_k} ; a_{k+1} is holomorphic on $\Pi_{r_{k+1}}$, continuous on $\overline{\Pi_{r_{k+1}}}$.

2) The map V_k is a biholomorphism of $\Pi_{r_{k+1}}$ onto its image.

Proof 1) The proof is by induction on k .

2) The map $(y + h_k, z)$ is one-to-one. Indeed, if $y_1 + h_k(y_1, z) = y_2 + h_k(y_2, z)$ for some points $(y_1, z), (y_2, z) \in \Pi_{r_{k+1}}$, then by (35) $|y_2 - y_1| = |h_k(y_1, z) - h_k(y_2, z)| \leq C_3' |y_2 - y_1|$ with $C_3' < 1$. ■

We may therefore consider the mapping $H_k = (y + h_0) \circ (y + h_1) \circ \dots \circ (y + h_k)$, which is a biholomorphism of $\Pi_{r_{k+1}}$ into Π_{r_0} .

Since $\|a_k\|_{r_k} \leq \delta_k^\beta$, we may apply Lemma 4 and get $\|D_y h_k\| \leq C_4 \delta_k$ on $\Pi_{r_{k+1}}$. It follows that the derivative of H_k satisfies the estimates

$$\max_{\overline{\Pi_{r_{k+1}}}} \|D_y H_k\| \leq (1 + C_4 \delta_0)(1 + C_4 \delta_1) \dots (1 + C_4 \delta_k) < e^{C_4 \Delta(\delta_0)} \tag{37}$$

The sequence $H_k(y, z)$ is Cauchy in the norm $\|\cdot\|_\rho$ on the domain Π_ρ . Indeed, by Remark 5 and (37)

$$\|H_k - H_{k+1}\|_\rho \leq e^{C_4\Delta(\delta_0)}\delta_{k+1}^{-\alpha}\|a_{k+1}\|_{r_{k+1}} \leq e^{C_4\Delta(\delta_0)}\delta_{k+1}^{\beta-\alpha}$$

and the series $\sum \delta_{k+1}^{\beta-\alpha}$ converges. Using Remark 4, H_k converges to an analytic function H .

The limit H is one-to-one. The proof is the same as of Remark 6, since

$$\max_{\Pi_{r_{k+1}}} \|D_y H_k - I\| \leq (1 + C_4\delta_0)(1 + C_4\delta_1)\dots(1 + C_4\delta_k) - 1 < e^{C_4\Delta(\delta_0)} - 1 < \frac{1}{2}$$

Finally, the substitution $v = H(y, z)$ reduces the system (30) to its linear part because $a_k \rightarrow 0$ (since $\|a_k\|_\rho \leq \delta_k^N$).

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