My project was in part to investigate the variance of playing the Shepp urn under the strategy which maximizes expected value. To this end after much experimentation I was able to conjecture and subsequently prove that if we play the Shepp urn with \( p \) positive balls and \( \beta(p) \) negative balls then the variance of our results grows at a rate of roughly \( \sqrt{p} \). I didn’t see as obvious of a pattern for other starting values, but the same general ideas may work there as well.

We recall the basic definitions of a Shepp Urn. A Shepp Urn is an optimal stopping time game in which a given urn contains \( m \) “minus” balls and \( p \) “plus” balls. The player at each stage may choose to either stop playing, or draw another ball at random from the urn, without replacement. When the player stops, s/he receives \( p^* - m^* \) dollars where \( p^* \) and \( m^* \) are the number of plus and minus balls drawn respectively. Let \( V(m, p) \) denote the expected value of a Shepp Urn game of an urn containing \( m \) - balls and \( p \) + balls Larry Shepp observed that the play which optimizes expected value of the game is to draw until the number of minus balls remaining is greater than \( \beta \) of the number of plus balls remaining. However often it is interesting not only to know the expected value of a game, but also its variance.

**Definition 1.** \( \beta(p) := \max\{m \in \mathbb{N} | V(m, p) > 0\} \)

Larry Shepp showed in [3] that optimal play is to draw from the urn until the number of minus balls remaining is greater than \( \beta \) of the number of plus balls remaining. However often it is interesting not only to know the expected value of a game, but also its variance.

**Definition 2.** Let \( U(m, p) \) be the random variable given by a game of drawing from a Shepp Urn with \( m \) ”minus” balls and \( p \) ”plus” balls under optimal play as defined by Shepp. For brevity of notation let \( U_p := U(\beta(p), p) \) as this is the game we will most often consider.

Given the definitions we state the goal theorem of the project:

**Theorem 1.** \( \Var(U_p) = \Var(U(\beta(p), p)) = \Theta(\sqrt{p}) \)

The first step in the proof of this theorem is to provide the upper bound. To do this we need the following little lemma:

**Lemma 2.** \( U_p \in [\beta(p) - p, 1] \)
Proof. First we prove that $U_p \leq 1$. To do this it is enough to show that we stop drawing if we are ever up by a dollar. This is equivalent to saying that $V(\beta(p) - t, p - t - 1) < 0$ for all $t \geq 0$. This follows from Boyce’s result in [1] that $V(m, p) \leq V(m + k, p + k)$ for all $k \geq 0$ as

$$V(\beta(p) - t, p - t - 1) \leq V(\beta(p) + 1, p) < 0$$

For the lower bound we simply note that a player at any point in the game can always guarantee at least $\beta(p) - p$ earnings out of the urn game $U(\beta(p), p)$ simply by drawing all of the balls remaining in the urn. Therefore $U_p \geq \beta(p) - p$.

Now we are in a position to quickly prove an upper bound on $\text{Var}(U_p)$.

**Lemma 3.** Let $X$ be a random variable such that $\mathbb{E}(X) = 0$ and for some $a, b \geq 0$ $-a \leq X \leq b$. Then $\text{Var}(X) \leq ab$.

**Proof.** Define the random variable $Y := \frac{X + a}{a + b}$. We note that $Y \in [0, 1]$ and $\text{Var}(Y) = \frac{\text{Var}(X)}{(a+b)^2}$. Let $\mu = \mathbb{E}(Y) = \frac{a}{a+b}$. We can compute that

$$\text{Var}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 \leq \mathbb{E}(Y) - \mathbb{E}(Y)^2 = \mathbb{E}(Y)(1 - \mathbb{E}(Y)) = \frac{a}{a+b} \cdot \frac{b}{a+b}$$

So we immediately have $\text{Var}(X) \leq ab$. \qed

**Corollary 4.** $\text{Var}(U_p) = O(\sqrt{p})$

**Proof.** Define $X := U_p - V(\beta(p), p)$. We see $\mathbb{E}(X) = 0$ and by Lemma 2 we have $-(\beta(p) - p + V(\beta(p), p)) \leq X \leq 1 - V(\beta(p), p)$. Combining these with the observation that $V(\beta(p), p) \in (0, 1]$ we can apply the above lemma to conclude that

$$\text{Var}(U_p) = \text{Var}(X) \leq [\beta(p) - p + V(\beta(p), p)][1 - V(\beta(p))] \leq \beta(p) - p + 1$$

By Shepp’s article [3] we know that $\beta(p) - p \sim \alpha\sqrt{2p}$ where $\alpha = 0.83992\ldots$ and so we therefore have $\text{Var}(U_p) = O(\sqrt{p})$. \qed

Now we try our hand at proving the slightly trickier bound that $\text{Var}(U_p) = \Omega(\sqrt{p})$. To do this we will need a few lemmas. First we will prove a sort of convexity result about the set of Shepp urns which are playable.

**Lemma 5.** Let $U(m, p, w)$ denote the expected value of the game of Shepp Urn played by inserting $m$ minus balls, $p$ plus balls, and $w$ worthless balls, which do not affect the score. Then $U(m, p, w) = V(m, p)$. 2
**Proof.** We prove this intuitively obvious fact by induction on the number of balls in the urn. If there is only one ball in the urn then the result is obvious. For the inductive step we check

\[
U(m, p, w) = \frac{m}{m + p + w} (U(m - 1, p, w) - 1) + \frac{p}{m + p + w} (U(m, p - 1, w) + 1) + \frac{w}{m + p + w} U(m, p, w - 1)
\]

\[
= \frac{m + p + w}{m + p} \left( \frac{m}{m + p} (V(m - 1, p) - 1) + \frac{p}{m + p} (V(m, p - 1) + 1) + \frac{w}{m + p} V(m, p) \right)
\]

\[
= \frac{m + p}{m + p + w} \left( V(m, p) + \frac{w}{m + p} V(m, p) \right) = V(m, p)
\]

Now we are ready to prove our convexity lemma:

**Lemma 6.** If \( V(m, p) > 0 \) then for any \( t \in [0, 1] \) then \( V([tm], [tp]) \geq tV(m, p) \).

**Proof.** Consider the following game, let \( w := m + p - [tm] - [tp] \). We play the game \( U([tm], [tp], w) \) but paint artificial plusses on \([tp]\) of the worthless balls and write minus on the other \([tm]\) worthless balls. By the previous Lemma this game will still have value \( V([tm], [tp]) \). However now we can play it with the following strategy: Ignore whether a ball is a scoring ball or a worthless ball, and play the game following the optimal strategy for the Shepp urn with \( m \) minus balls and \( p \) plus balls. This strategy will have expected return at most \( V([tm], [tp]) \). However a moments consideration reveals that by linearity of expectation this game will also have expected return at least \( tV(m, p) \).

Let \( X_i \) denote the draw of the \( i \)th ball (with value 0 if it is not drawn), and \( A_i \) the event that the \( i \)th ball is drawn and is positive, and \( B_i \) the event that the ball is drawn and is negative. If we let \( p_i \) be the probability that \( X_i \) is drawn, and \( q_i \) the probability that \( X_i \) is drawn, given we are playing under the expectation maximising strategy for \( V(m, p) \). Note that the probability that a given positive ball scores is \( \frac{[tp]}{p} \geq t \) and likewise the probability a negative ball scores is \( \frac{[tm]}{m} \leq t \). Then we have by linearity of expectation that the expectation of playing this game given this strategy is

\[
E \left( \sum_{i=1}^{m+n} X_i \right) = \sum_i E(X_i) \geq \sum_i tp_i - tq_i = tV(m, p)
\]

So therefore we have our result. \[\square\]

Note that this result implies that if an urn with \( m \) minus and \( p \) plus balls is playable with positive expected value, than for any \( p' < p \) the urn with \( p' \) plus balls and \( \left\lfloor \frac{m}{p'} \right\rfloor \) minus balls is playable. Some experimental checking revealed that contrary to my initial
expectations, we do not have that $V(m, p) > 0$ and $V(m', p') > 0$ implies that $V(tm + (1 - t)m', tp + (1 - t)p') > tV(m, p) + (1 - t)V(m', p')$. However I do not yet know whether we can say that $V(tm + (1 - t)m', tp + (1 - t)p') > 0$. Now unto another crucial lemma which provides a lower bound for ballot counting problems of the following form: If in an election between two candidates the winner has received $n$ votes and the loser $m$ where $n > \alpha m$, then what is the probability that if the votes are counted in an arbitrary order, then at all points in the counting the winner has received at least $\alpha$ times as many votes as the loser. Note that when we have $\alpha = 1$ and $n = m$ then this is exactly analagous to the catalan number $C_n$. Here we provide a lower bound to this probability.

Lemma 7. Let $\alpha \in \mathbb{R}^+$ and $m, n \in \mathbb{N}$ such that $n - \alpha m \geq 0$. Then if we take $\sigma$ to be a random ordering of $m$ 0’s and $n$ 1’s (given by a map from $[m+n]$ to the multiset $\{0^m, 1^n\}$), then with probability at least $\frac{n - \alpha m}{n + m}$ we have that for all $1 \leq t \leq m + n$ that

\[
\frac{\alpha}{\alpha + 1} t \leq \sum_{j=1}^{t} \sigma(j)
\]

Proof. Take such a permutation $\sigma$, and consider the action of $k \in \mathbb{Z}_{m+n}$ on $\sigma$ by $[k\sigma](t) := \sigma(t + k)$. Informally stated, this action takes the a written permutation, and shifts all of the entries $k$ spaces to the right, with the last $k$ entries wrapping around to the beginning of the permutation. This action splits the space of permutations into orbits of size at most $m + n$ and so it suffices to show that in each orbit there are at least $\frac{n - \alpha m}{n + m}$ permutations satisfying the required condition that for all $1 \leq t \leq m + n$ we have $\frac{\alpha}{\alpha + 1} t \leq \sum_{j=1}^{t} \sigma(j)$.

For this define the helper functions

\[
\theta(i) := \begin{cases} 1 & \text{if } i = 1 \\ -\alpha & \text{if } i = 0 \end{cases}
\]

\[
a_{\sigma}(t) := \sum_{j=1}^{t} \theta(\sigma(j))
\]

We can quickly check that (1) is satisfied if and only if $a_{\sigma}(t) \geq 0$. We now recursively find a sequence of at least $\lfloor n - \alpha m \rfloor + 1$ rotations such that $k\sigma$ satisfies (1). Let $C := \min_{t \in [m+n]} a(t)$, and define the set

\[
A_{\sigma} := \{ t \text{ s.t. } 0 \leq t \leq m + n, \text{ and } a_{\sigma}(t) \leq C + n - \alpha m \text{ and } \forall s \geq t a_{\sigma}(t) \leq a_{\sigma}(s) \}
\]

$A_{\sigma}$ will consist of all points that reading $(a_{\sigma}(t))_{t=0}^{m+n}$ from right to left yield the lowest value so far (so long as that value is sufficiently low). First we note that for all $t \in A_{\sigma}$ we have $t\sigma$ satisfies condition (1). This follows from noting that for $s \leq m + n - t$ we have
\[ a_{t\sigma}(s) = a_{\sigma}(t + s) - a_{\sigma}(t) \geq 0 \text{ And for } s \geq m + n - t \]

\[ a_{t\sigma}(s) = \sum_{j \notin [s-(m+n-t)] \cup [t,m+n]} \theta(\sigma(j)) = a_{\sigma}(m + n) - a_{\sigma}(t) + a_{\sigma}(s - (m + n - t)) \]

\[ \geq n - \alpha m - (C + n - \alpha m) + \alpha_{\sigma}(s - (m + n - t)) \geq 0 \]

This verifies that for all \( t \in A_\sigma \), \( t \sigma \) satisfies (1).

Now to complete the proof we just need to show that \(|A_\sigma| \geq |n - \alpha m|\). To see this order the elements of \( A_\sigma \) and label them \( t_1, t_2, \ldots, t_k \). We first note that \( a_{\sigma}(t_1) = C \). Further \( a_{\sigma}(t_{i+1}) \leq a_{\sigma}(t_i) + 1 \) for all \( i \) as \( a_{\sigma}(t_i + 1) = a_{\sigma}(t_i) + 1 \). From this observation and the definition of \( A_\sigma \) we can see that we must also have \( a_{\sigma}(t_k) > C + n - \alpha m - 1 \). So combining these facts we see that

\[ k - 1 \geq a_{\sigma}(t_k) - a_{\sigma}(t_1) > n - \alpha m - 1 \]

So because \( k \) is an integer \(|A_\sigma| = k \geq |n - \alpha m| + 1 \) which completes the proof. \( \square \)

**Lemma 8.** Let \( X_i \) denote the \( i^{th} \) ball drawn from an urn containing \( \beta(p) - 1 \) balls and \( p + 1 \) balls. Then with probability \( \Omega(1) \) we have that \( Y := \sum_{i=1}^{\beta(p)+p/2} X_i \leq p - \beta(p) \)

**Proof.** Note if we let \( Z_i \) be the same drawing, but now with 0 balls instead of \(-1\) balls, and \( Z = \sum_{i=1}^{\beta(p)+p/2} Z_i \) then we have \( Y = \sum_{i=1}^{\beta(p)+p/2} 2Z_i - 1 \) So it is enough to show that

\[ Z = \sum_{i=1}^{\beta(p)+p/2} Z_i \leq \frac{1}{2} \left( \frac{\beta(p) + p}{2} + p - \beta(p) \right) = \frac{p}{2} + \frac{p - \beta(p)}{4} = \mathbb{E}(Z) - \frac{\beta(p) - p}{4} \]

with probability \( \Omega(1) \). But \( \beta(p) - p = O(\sqrt{p}) \) and \( Z \) is a standard hypergeometric random variable. But this is an example specifically considered in \([2]\) where it is proved that

\[ \sigma^2 \equiv \text{Var}(Z) = \left( \frac{p + \beta(p)}{p + \beta(p) - 1} \right) \frac{\beta(p)}{p + \beta(p)} \frac{p}{2} = \frac{\beta(p)p}{4(p + \beta(p) - 1)} \sim \frac{p}{8} \]

and that by a Central Limit Theorem type result we have \( Pr(Z - \mathbb{E}(Z) \leq \frac{\beta(p) - p}{4}) = \Omega(1) \)

\[ \lim_{p \to \infty} Pr \left( Z - \mathbb{E}(Z) < \frac{p - \beta(p)}{4} \right) = \int_{-\infty}^{\beta(p)/2} e^{-t^2/2} dt = \int_{-\infty}^{\beta(p)/4} e^{-t^2/2} dt = \Omega(1) \]

\( \square \)

**Lemma 9.** \( \beta(p) - p \) is an increasing function of \( p \)
Proof. Let $p < r$. It is enough to show that $V(r, r + \beta(p) - p) > 0$. But this follows from Boyce's observation (insert citation) that $V(a + k, b + k) \geq V(a, b)$. So here we have

$$V(r, r + \beta(p) - p) = V(p + (r - p), \beta(p) + (r - p)) \geq V(p, \beta(p)) > 0$$

\(\square\)

**Lemma 10.** If at any point while playing the urn game $U_p$ there are less than $\beta(t)$ minus balls remaining in the urn, then the game ends in a loss of at least $\beta(p) - p - (\beta(t) - t) - 1$ dollars.

**Proof.** Say the game ends with $m^*$ minus balls and $p^*$ plus balls remaining in the urn. First we note that the last ball drawn was a plus ball \[1\] and drew from the urn with $p^* + 1$ balls so $m^* \leq \beta(p^* + 1)$. Because we stopped playing we must have that $\beta(p^*) < m^* \leq \beta(t)$ and so we have $p^* < t$ and in particular $p^* + 1 \leq t$. Now we can combine the above with Lemma 9 to compute the value of the game played to be

$$(p - p^*) - (\beta(p) - m^*) \leq 1 + (p - \beta(p)) - (p^* + 1 - \beta(p^* + 1))$$

$$\leq 1 + (p - \beta(p)) - (t - \beta(t))$$

\(\square\)

**Lemma 11.** With probability $\Omega \left( \frac{1}{\sqrt{p}} \right)$ the game $U_p$ ends in a loss of at least $(1 + o(1))(\alpha - \frac{\alpha}{\sqrt{2}})\sqrt{p}$ dollars.

**Proof.** A game of Shepp Urn can be simulated by generating a random permutation of the balls representing the order in which they are drawn, given by a map $\sigma : \{(-1)^{\beta(p)}, 1^p\} \rightarrow [p + \beta(p)]$. We note that because of the preceding lemma the probability that the game ends with such a loss can be lower bounded by the probability that at least $\beta(p)$ minus balls are drawn. We also note that by our convexity lemma, $\beta \left( \frac{p^2}{2} \right) \geq \beta(p)$ and so if we let $t$ denote the least integer such that $\beta(t) > \beta(p)/2$ then we have

$$\beta(p) - p - \left( \frac{\beta(p)}{2} - t \right) - 1 \geq \beta(p) - p - \left( \frac{\beta(p)}{2} - \frac{p}{2} \right) - 1 \sim \alpha \sqrt{\beta(p)} - \alpha \sqrt{\frac{p}{2}}$$

$$= \left( \alpha - \frac{\alpha}{\sqrt{2}} \right) \sqrt{p}$$

We can also lower bound the probability that after $\beta(p)/2 + p$ draws there have been at least $\beta(p)/2 + \sqrt{p}$ minus balls drawn and the game has not stopped. Let $\hat{p}$ and $\hat{m}$ denote the number of positive and minus balls drawn from the urn after $\beta(p)/2$ draws. The event that $\hat{m} > \beta(p)/2 + \sqrt{p}$ occurs with probability $\Omega(1)$ by lemma 8. To check if the game is still
being played, we note that it is enough to check that for all draws up to the \( \frac{\beta(p) + p}{2} \) draw if \( m^* \) and \( p^* \) are the number of negative and positive balls drawn respectively then at we have \( \beta(p - p^*) < \beta(p) - m^* \). By our convexity lemma we see that this is certainly satisfied so long as \( \beta(p) - m^* \leq \frac{\beta(p)}{p}(p - p^*) \). Next one checks that this condition is equivalent to \( m^* \geq \frac{\beta(p)}{p}(1/2) + \sqrt{p} \). By our convexity lemma we see that this is certainly satisfied so long as \( \beta(p) - m^* \leq \frac{\beta(p)}{p}(1/2) \). But this is exactly the sort of ballot counting problem considered in lemma 7. So from that result, given that \( \hat{m} > \frac{\beta(p)}{2} + \sqrt{p} \) the probability that the game is still being played is at least

\[
\frac{|\hat{m} - \frac{\beta(p)}{p} \hat{p}|}{\frac{\beta(p) + p}{2}} \geq \frac{\beta(p) \frac{2}{p} + \sqrt{p} - \frac{\beta(p)}{p} (\frac{p}{2})}{\frac{\beta(p) + p}{2}} = \frac{2 \sqrt{p} \beta(p) + p}{\beta(p) + p} \sim \frac{1}{\sqrt{p}}
\]

So we can conclude that with probability \( \Omega \left( \frac{1}{\sqrt{p}} \right) \) the game is still being played after \( \frac{\beta(p) + p}{2} \) draws and that \( \hat{m} > \frac{\beta(p)}{2} + \sqrt{p} \). So we a fortiori have our result that the game is still being played after \( \frac{\beta(p) + p}{2} \) draws with probability \( \Omega \left( \frac{1}{\sqrt{p}} \right) \) and so with probability \( \Omega \left( \frac{1}{\sqrt{p}} \right) \) the game ends in a loss of at least \((1 + o(1))(\alpha - \frac{\alpha}{\sqrt{2}})\sqrt{p}\) dollars.

**Corollary 12.** \( \text{Var}(U_p) = \Omega(\sqrt{p}) \)

**Proof.**

\[
\text{Var}(U_p) \geq \Pr \left[ E(U_p) - U_p \geq (1 + o(1))(\alpha - \frac{\alpha}{\sqrt{2}})\sqrt{p} \right] \geq \left( (1 + o(1))(\alpha - \frac{\alpha}{\sqrt{2}})\sqrt{p} \right)^2 = \Omega \left( \frac{1}{\sqrt{p}} \right) \cdot [\Omega(\sqrt{p})]^2 = \Omega(\sqrt{p})
\]

Combining this corollary with Corollary 4 we have our desired result for the project that \( \text{Var}(U_p) = \Theta(\sqrt{p}) \).

**References**

