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Research Statement

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My research is mainly in Analytic Number Theory with attention currently directed to Automorphic Forms and the Quantum Unique Ergodicity Conjecture. Quantum unique ergodicity of Hecke-Maass cusp forms on $\Gamma\backslash\mathbb{H}$, with $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, is particularly interesting due to its physical and mechanical implications.

Quantum Unique Ergodicity

Let $\mathbb{H}$ be the upper half plane with hyperbolic measure $d\mu z := y^{-2}\, dx\, dy$. Set $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and let $X = \Gamma\backslash\mathbb{H}$ be the quotient space on which we define the Laplace-Beltrami operator

$$\Delta := y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

Denote by $\mathcal{L}(X)$ the Hilbert space of square integrable automorphic forms with inner product

$$< f, g > := \int_X f(z)\, \bar{g}(z) d\mu z.$$

The discrete spectrum of X consists of the so called Hecke-Maass cusp forms. These forms ϕ are square integrable eigenfunctions of Δ and simultaneous eigenfunctions of the normalized Hecke operators T_n

$$\Delta\phi + \lambda\phi = 0, \quad \lambda = s(1-s), \quad s = \frac{1}{2} + it$$

$$T_n \phi = \lambda_\phi(n)\, \phi \quad \text{for all} \quad n \geqslant 1.$$

For convenience, we normalize these cusp forms so that $\int_X |\phi(z)|^2 d\mu z = 1$.

Taking an orthonormal basis $\{\phi_j\}$ for the subspace of cusp forms $\mathcal{C}(X)$ and ordering them by their eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$, we can state the Quantum Unique Ergodicity Conjecture as follows:

Conjecture(Quantum Unique Ergodicity):
Define the probability measures $\mu_j := |\phi_j(z)|^2 d\mu z$*, then* $\mu_j \to d\mu z/\mathrm{Vol}(X)$ *as* $j \to \infty$.

The principle attraction behind this conjecture is that relations between classical and quantum mechanics suggest that this limit would **not** exist. For brevity, I will not go into full detail about the mechanics behind the problem, but will instead mention a couple **unexpected** properties exhibited in our case. Those interested in further familiarizing themselves with the subject matter are recommended to read a survey by Sarnak[Sa2] and exposition by Gutzwiller[Gu]. From now on, the measures μ_j and forms ϕ_j will be as defined above.

Quantum Chaos

Quantum chaos is concerned with the relation between classically chaotic systems and their quantum models. Taking a classical system and relating it to a quantum model is known as a quantization. If a system is classically regular, then similar behavior should be exhibited on the quantum level. The same holds for chaotic systems.

By a quantization due to Zelditch[Ze], the eigenfunctions ϕ_j are realized to be the allowed quantum energy states of a classically chaotic system, geodesic flow on a manifold of negative curvature. What we expect, therefore, is some recognition of chaos on the quantum level. For example, we expect local spacing statistics for the eigenvalues λ_j to resemble those of the eigenvalues of a random real symmetric matrix of large size[Me]. Consecutive eigenvalues of such a matrix seem to remain fairly distanced from one another.

However, numerical experiments have shown that local spacing statistics of our eigenvalues λ_j appear to be "Poissonian"[Sa2]: Setting $\tilde{\lambda_j} = \lambda_j/12$, then for $0 \leqslant \alpha < \beta < \infty$,

$$\frac{\#\{j \leqslant N \mid \tilde{\lambda}_{j+1} - \tilde{\lambda}_j \in [\alpha, \beta]\}}{N} \longrightarrow \int_\alpha^\beta e^{-x} dx \quad \text{as} \quad N \longrightarrow \infty.$$

Therefore, these eigenvalues do not seem to repel one another and are the first sign of peculiar behavior in our quantum model. Similarly, one would expect a subsequence of eigenstates to behave "badly" by having large values localize to certain geodesics on X. This phenomenon is referred to as "scarring" and is often seen in chaotic systems. Quantum unique ergodicity is a statement against such localization of mass.

Main Results

Motivated by a common "unfolding" technique, for $\lambda_j(n)$ the Hecke normalized Fourier coefficients of the Hecke-Maass cusp forms ϕ_j, we reduce the Quantum Unique Ergodicity Conjecture to the following theorem:

Theorem 1. *The Quantum Unique Ergodicity Conjecture is equivalent to the following: For any integer $h \neq 0$ and any $G \in C_0^\infty(0, \infty)$*

$$\sum_n \lambda_j(n)\, \lambda_j(n+h) G(\frac{n}{t_j}) = o\,(t_j\, L(1, \mathrm{sym}^2 \phi_j)) \qquad (1)$$

as $t_j \longrightarrow \infty$.

We also require the proper asymptotic formula for $h = 0$, but this is immediately obtained from a standard Rankin-Selberg function method.

Sums of type

$$\sum_n \lambda_j(n)\, \lambda_j(n+h) \qquad (2)$$

are called shifted convolution sums and are the main focus of my research. Having these sums arrise in such a natural problem is the motivation behind studying their properties. One approach towards (1) would be to look for cancellations among the summation terms in (2). However, results in my research suggest that the true bound on the average size of the coefficients could eventually lead to a proof of quantum unique ergodicity. This true bound would be very exciting for several reasons.

The Ramanujan-Petersson conjecture asserts that $|\lambda_j(p)| \leqslant 2$ for prime p. Thus $|\lambda_j(n)| \leqslant \tau(n)$ for any integer n. Here τ is the divisor function and this bound is best possible. The $\lambda_j(n)$, therefore, are not uniformly bounded. However, primes p for which $|\lambda_j(p)| > 1$ are very rare. Indeed, by Rankin-Selberg theory we know that

$$\sum_{n \leqslant X} |\lambda_j(n)|^2 \ll X. \qquad (3)$$

The Sato-Tate conjecture provides the true density function for the distribution of values $\lambda_j(p)$. One can show[E-M-S] that

$$\sum_{n \leqslant X} |\lambda_j(n)| \ll \frac{X}{(\log X)^{1-8/3\pi}}.$$

Therefore, the terms $\lambda_j(n)$ are relatively small quite often and large values occur very rarely.

In the case of the shifted convolution sum with $h \neq 0$, we now try to estimate the size of

$$\sum_{n \leqslant X} |\lambda_j(n)\lambda_j(n+h)|. \tag{4}$$

With this shift, however, there is one new important factor to consider. Could it be possible, for some values of h, that the n-th and $n+h$-th terms "conspire" against each other so that this sum would be large(as large as X)?

Consider the bound needed in Theorem 1. Without taking absolute values, we are already requiring $\sum_{n \sim t_j} \lambda_j(n)\lambda_j(n+h)$ to grow slower than $t_j\, L(1, \mathrm{sym}^2 \phi_j)$ as $t_j \longrightarrow \infty$. By the Grand Riemann Hypothesis, we expect the values of $L(1, \mathrm{sym}^2 \phi_j)$ to fluctuate in a range of

$$\frac{1}{\log\log t_j} \ll L(1, \mathrm{sym}^2 \phi_j) \ll \log\log t_j.$$

The worst scenario, under this hypothesis, therefore demands

$$\sum_{n \sim t_j} \lambda_j(n)\lambda_j(n+h) = o\left(\frac{t_j}{\log\log t_j}\right) \quad \text{as} \quad t_j \longrightarrow \infty \tag{5}$$

to hold true in order to achieve quantum unique ergodicity. In other words, a uniform bound for all ϕ_j of any power $\delta \geqslant 0$ saving of $\log t_j$

$$\sum_{n \sim t_j} \lambda_j(n)\lambda_j(n+h) \ll \frac{t_j}{(\log t_j)^\delta}$$

would give us the desired result.

In my sum (4), however, I am dealing only with positive terms in the summation. Using Seive Theory, we exploit the positivity of the terms and obtain

Theorem 2. *For a Hecke-Maass cusp form ϕ_j, integer $h \neq 0$ and $X \asymp t_j$ we have*

$$\sum_{n \leqslant X} |\lambda_j(n)\lambda_j(n+h)| \ll_{h,t_j} \frac{X}{(\log X)^\delta}$$

for some absolute positive constant δ.

This result depends on the assumption of the Ramanujan-Petersson Conjecture, but this assumption is not made until the very end of the proof after an upper bound sieve is applied. During the application of the upper bound sieve, it was observed that the extra variable $h \neq 0$ itself was what actually provided the necessary independence for the success of the proof. In fact a similar method, would give the same if not better bound than in Theorem 2 for the more complicated sums

$$\sum_{n \leqslant X} |\lambda_j(n)\lambda_j(n+h_1)\lambda_j(n+h_2)...\lambda_j(n+h_k)| \tag{6}$$

where each of the h_i's are distinct. The method developed for this case, however, still has room for improvement and is currently work in progress.

Final Remarks

Theorem 2 is a statement for one Hecke-Maass cusp form and does not imply a uniform bound for all ϕ_j in the basis. However, there is still hope for progress in this direction. Uniformity will be achieved with the appropriate bounds on symmetric square and symmetric fourth L-functions near the point $s = 1$. Given the opportunity, I would like to continue studying shifted convolution sums in the hopes of developing other techniques to obtain better bounds in the direction of quantum unique ergodicity.

References

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