On What We Don’t Know (About List Coloring)

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There’s “normal” graph coloring:

**Def.** A graph is *k-colorable* if there is a function $c : V(G) \rightarrow \{1, \ldots, k\}$ such that

$$v \sim w \Rightarrow c(v) \neq c(w)$$

Then we can define the *chromatic number* as

$$\chi(G) = \min\{k \mid G \text{ is } k\text{-colorable.}\}$$
Then there is *list coloring*:

**Def.** A *list assignment* for a graph $G$ is an assignment of a list $L_v$ (usually a subset of $\mathbb{N}$) to each vertex $v \in G$.

Let

$$\mathcal{L} = \{L_v \mid v \in V(G)\}$$

and we define the *palette* as

$$P_{\mathcal{L}} = \bigcup_{v \in V(G)} L_v$$

We then say that $G$ is $\mathcal{L}$-choosable if there is a function $c : V(G) \to P_{\mathcal{L}}$ such that

$$v \sim w \Rightarrow c(v) \neq c(w) \text{ and } c(v) \in L_v, c(w) \in L_w$$
**Def.** For a function $f : V(G) \rightarrow \mathbb{N}$, we say that $G$ is $f$-choosable if for any list assignment $\mathcal{L}$ satisfying $|L_v| = f(v)$ for all $v \in V(G)$, $G$ is $\mathcal{L}$-choosable.

If $f \equiv k$ is a constant function, then we say that $G$ is $k$-choosable and say that $\chi_l(G) = k$.

Most of the interest so far in list coloring has dealt with $k$-choosability.
List Coloring is Different!

\[ \chi_l(G) \geq \chi(G) \] since “normal” coloring is equivalent to assigning the same list of colors to each vertex in the graph. However, notice:
The First Example, Always, With List Coloring

This list assignment shows that $\chi_l(K_{3,3}) = 3 \neq \chi(K_{3,3})$. 
More Generally...

Fact. \[ \chi_l \left( K_{\binom{2n-1}{n}, \binom{2n-1}{n}} \right) = n + 1 \]

Proof. We assign as lists on each side the \( n \)-subsets of \( \{1, 2, \ldots, 2n - 1\} \). Then we can color if and only if we use only \( n - 1 \) colors on one side. However, for each choice of \( n - 1 \) colors there is a vertex that misses precisely those colors, and hence can’t be colored.

Consequence: In general we cannot say anything about \( \chi_l(G) \) given \( \chi(G) \).
Conclusions for Planar Graphs

**Theorem [Thomassen 1993]:** Every planar graph is 5-choosable.

**Theorem [Voigt 1993]:** There are planar graphs that are not 4-choosable.

Voigt’s example had

The smallest-known example of a non-4-choosable planar graph has 75 vertices [Gutner 1996].
What’s Different About List Coloring?

There are some obvious statements about “normal” coloring whose list-coloring counterparts aren’t so obvious. For example,

**Obvious Fact.** If $\chi(G) = t$ and $s < t$, then there is a subgraph $H \subseteq G$ such that

$$|V(H)| \geq \frac{s}{t}|V(G)|$$

and $\chi(H) = s$.

**Proof.** Color $G$ with $t$ colors and select the $s$ largest color classes as $H$. 
Conjecture 1: Albertson, Haas, Grossman [2000]

If $\chi_l(G) = t$ and $\mathcal{L}$ is a family of assignments where each vertex is assigned a list $L_v$ of $s$ colors ($s < t$), then there is a subgraph $H \subseteq G$ such that

$$|V(H)| \geq \frac{s}{t}|V(G)|$$

and $H$ is $\mathcal{L}$-choosable.

Note: The more direct analogue is not true: there are graphs $G$ with $\chi_l(G) = t$ and $s < t$ such that there are no subgraphs $H \subseteq G$ with $\chi_l(H) = s$ satisfying

$$|V(H)| \geq \frac{s}{t}|V(G)|$$
Progress on Conjecture 1

**Theorem:** If $s|t$, then the conjecture is true.

**Proof:** For sake of clarity, let $s = 2$ and $t = 4$. Each vertex $v \in G$ is given a list of two colors $L_v = \{a_v, b_v\}$. Append doppelgänger colors $a'_v$ and $b'_v$ to each list, so each new list is $L'_v = \{a_v, b_v, a'_v, b'_v\}$. If $\mathcal{L}'$ is the family of new lists, then $G$ is $\mathcal{L}'$-choosable.
Progress of Conjecture 1 (Continued)

Color $G$ using $\mathcal{L}'$.

Now, for each color $c$ in the palette, some vertices may have been colored $c$ and some may have been colored $c'$. Let $V_c$ be the bigger of those two sets of vertices. Finally, let

$$H = \bigcup_{c \in \mathcal{P}_\mathcal{L}} V_c$$

and notice that each vertex in $H$ colored by a doppelgänger can be re-colored with its original color.
Theorem [Chappell 1999]: If $\chi_l(G) = t$ and $s < t$ then there is a subgraph $H$ with the required properties such that

$$|V(H)| \geq \frac{6}{7} \left( \frac{s}{t} |V(G)| \right)$$

Chappell’s proof is based on simple probabilistic arguments.

The rest of the conjecture is still wide open. Even the case of $s = 2, t = 3$ remains a mystery.
Another Direction: Graphs where $\chi_l(G) = \chi(G)$.

The following graphs are known to satisfy $\chi_l(G) = \chi(G)$:

- (Galvin 1995) Line graphs of bipartite graphs.
- (Gravier, Maffray 1995) Complements of triangle-free graphs.
- (Ohba 2001) Graphs satisfying $|V(G)| \leq \chi(G) + \sqrt{2\chi(G)}$.
- (Reed, Sudakov 2005) Graphs satisfying $|V(G)| \leq \frac{5}{3}\chi(G) - \frac{4}{3}$. 
Hard Conjecture Number 1

**Conjecture [Vizing 1976]:** Every line graph satisfies $\chi_l(G) = \chi(G)$.

This conjecture is important enough to be called *The List Coloring Conjecture.*
Conjecture [Gravier, Maffray 1997]: Every claw-free graph satisfies $\chi_I(G) = \chi(G)$.

Note that this conjecture is more general than hard conjecture number 1, and many people believe it is so general as to actually be false.
**Ohba’s Conjecture**

**Conjecture [Ohba 2001]:** If \( |V(G)| \leq 2\chi(G) + 1 \) then \( \chi_l(G) = \chi(G) \).

For Ohba’s Conjecture it suffices to consider only complete partite graphs where equality holds.
Complete Partite Graph Notation

**Definition:** $K(a_1, a_2, \ldots, a_k)$ is the complete $k$-partite graph with $a_i$ vertices in part $i$. Usually we write it so $a_1 \geq a_2 \geq \cdots \geq a_k$. If there are repetitions, we also write as shorthand

$$K(a_1 \ast n_1, a_2 \ast n_2, \ldots, a_k \ast n_k)$$
**Complete Partite Graph Example**

So, for example, the following graph is $K(3, 3, 1) = K(3 \ast 2, 1)$:

![Graph Diagram]

**Motivation**: The graph $G = K(4, 2 \ast (k - 1))$ satisfies $\chi(G) = k$, $|V(G)| = 2k + 2$, and $\chi_l(G) = k + 1$ iff $k$ is even!
Progress Towards Ohba’s Conjecture

Graphs for which Ohba’s Conjecture is true:

- (Erdős, Rubin, Taylor 1979) $K(2 \ast k)$.
- (Gravier, Maffray 1998) $K(3, 3, 2 \ast (k - 2))$.
- (Enomoto, Ohba, Ota, Sakamoto 2002) $K(4, 2 \ast (k - 2), 1)$.
- (Cranston 2007) $G$ such that $\alpha(G) = 3$, or $G$ with one part of size 4.
- (Shen, He, Zheng, Wang, Zhang 2007) $K(5, 3, 2 \ast (k - 5), 1 \ast 3)$.
- (Enomoto, Ohba, Ota, Sakamoto 2002) $K(m, 2 \ast (k - s - 1), 1 \ast s)$ for $m \leq 2s - 1$. 
The following ideas are used heavily in the previous results:

1. **(Hall 1935)** If $G = (A, B)$ is a bipartite graph such that $|N(S)| \geq |S|$ for all $S \subseteq A$, then there is a matching that saturates $A$. 
2. **(Kierstead 2000)** Let $G$ be given with list assignment $\mathcal{L}$. Let $X$ be a maximal set of vertices so that

$$|L(X)| := \left| \bigcup_{v \in X} L_v \right| < |X|$$

Then if $X$ is $\mathcal{L}|_X$-choosable, then $G$ is $\mathcal{L}$-choosable.

3. **(Kierstead 2000, Reed, Sudakov 2001)** If $G$ is $\mathcal{L}$-choosable for all list assignments such that $|L_v| = k$ and $|\mathcal{P}| < |V(G)|$, then $\chi_l(G) \leq k$. 

Machinery (New)
Chappell’s result suggests that the conjecture of Albertson, et. al. is true.

**Ambiguous Philosophical Thought:** Most results concerning Ohba’s Conjecture rely on *heavy* case analysis. Can it be avoided?
Lemma. $K(4, 3, 1, 1)$ is 4-choosable.

Proof. From the machinery mentioned earlier, it suffices to consider when the palette has at most 8 colors. If that is the case, then there is a set $C$ of at least 4 colors such that for each color $c \in C$, there are at least two vertices in the 4-set that has $c$ in their list.

Case to Always Exclude: If there is a color that is shared by all the vertices of the 4-set or the 3-set, then use that color and you’re in a much easier situation.
**Case to Exclude:** If both singleton vertices have the same list of colors, and that list is also the same as some vertex in the 3-set, then we can color everything.

Now, take a color $c \in C$, and WLOG there are two vertices, $v_1$ and $v_2$, in the 3-set that have $c$ in their list. Since we’ve excluded the singleton lists being equal and equal to a vertex in the 3-set, there is a choice of colors to color the singletons so that the remaining two vertices in the 4-set and the 3-set still have two valid colors remaining. So - what’s left if $K(3, 2)$, which we know is 2-choosable.
Finally . . .

Thank you for listening!