MATH 350-02 **Solutions to problems from Chapter 6**

#1 The four given vectors form an orthogonal set of nonzero vectors. Therefore this set is linearly independent and, hence, is a basis for $\mathbf{R}^4$. Then if $v = (5,3,7,1), v_1 = (1,1,1,1), v_2 = (1,1,1,-3), v_3 = (1,-2,1,0), v_4 = (1,0,-1,0)$ we have

$$v = \sum_{i=1}^{4} \frac{< v, v_i >}{< v_i, v_i >} v_i =$$

$$\frac{16}{4} v_1 + \frac{12}{12} v_2 + \frac{6}{6} v_3 + \frac{-2}{2} v_4 =$$

$$4v_1 + v_2 + v_3 - v_4.$$

Section 6.1, #5: Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then $< x, y >= x_1\bar{y}_1 + ix_1\bar{y}_2 - ix_2\bar{y}_1 + 2x_2\bar{y}_2$ and so $< x, x >= ||x_1||^2 + 2||x_2||^2$. Properties (a), (b), (c) of the definition follow from properties of matrix multiplication and property (d) follows since $||x_1||^2, ||x_2||^2 \geq 0$..

Also $< (1-i, 2+3i), (2+i, 3-2i) >= (1-i)(2-i) + i(1-i)(3+2i) - i(2+3i)(2-i) + 2(2+3i)(3+2i) = 6 + 21i$.

Section 6.1 #8a: $< (1,1), (1,1) >= 0$ so condition (d) of the definition does not hold.

Section 6.1 #8b: Let $I = I_2$ denote the 2 by 2 identity matrix. Then $< 2I, I >= tr(2I + I) = tr(3I) = 6$ and $< I, I >= tr(2I) = 4$. Thus $< 2I, I > \neq 2 < I, I >$, so condition (b) does not hold. For that matter, (a) and (d) don't hold either.

Section 6.1 #9a: Let $\beta = \{u_1, ..., u_k\}$. Since β is a basis, we may write $x = \sum_{i=1}^{k} a_k u_k$ for some scalars $a_1, ..., a_i$. Then $< x, x >=< \sum_{i=1}^{i} a_k u_k, x >= \sum_{i=1}^{k} a_k < u_k, x >= 0$. Hence (by condition (d) in the definition) $x = 0$.

Section 6.1 #9b: Let $w = x - y$. Then for any $z \in \beta$ we have $< w, z >=< x, z > - < y, z >= 0$. Thus by part (a), $w = 0$ and so $x = y$.

Section 6.2, #2a: Let the given vectors be v_1, v_2, v_3. Thus $v_1 = (1,0,1), v_2 = (0,1,1), v_3 = (1,3,3)$. Then $\{w_1, w_2, w_3\}$ is an orthogonal basis where

$$w_1 = v_1 = (1,0,1),$$

$$w_2 = v_2 - \frac{< v_2, w_1 >}{< w_1, w_1 >} w_1 = (0,1,1) - \frac{1}{2}(1,0,1) = (-\frac{1}{2}, 1, \frac{1}{2}),$$

$$w_3 = v_3 - \frac{< v_3, w_1 >}{< w_1, w_1 >} w_1 - \frac{< v_3, w_2 >}{< w_2, w_2 >} w_2 =$$

$$(1,3,3) - \frac{4}{2}(1,0,1) - \frac{4}{\frac{6}{4}}(-\frac{1}{2},1,\frac{1}{2}) =$$

$$(\frac{1}{3},\frac{1}{3},-\frac{1}{3}).$$

Setting $u_i = \frac{w_i}{||w_i||}$ for each i gives an orthonormal basis $\beta = \{u_1, u_2, u_3\}$ where

$$u_1 = (\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}),$$

$$u_2 = (\frac{-\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}),$$

$$u_3 = (\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{-\sqrt{3}}{3}).$$

Finally, if $x = (1,1,2)$ the Fourier coefficients of x with respect to β are $< x, u_1 >= \frac{3\sqrt{2}}{2}, < x, u_2 >= \frac{\sqrt{6}}{2}, < x, u_3 >= 0$.

Section 6.2, #2f: Let the given vectors be v_1, v_2, v_3. Thus $v_1 = (1,-2,-1,3), v_2 = (3,6,3,-1), v_3 = (1,4,2,8)$. Then $\{w_1, w_2, w_3\}$ is an orthogonal basis where

$$w_1 = v_1 = (1,-2,-1,3),$$

$$w_2 = v_2 - \frac{< v_2, w_1 >}{< w_1, w_1 >} w_1 = (3,6,3,-1) - \frac{-15}{15}(1,-2,-1,3) = (4,4,2,2),$$

$$w_3 = v_3 - \frac{< v_3, w_1 >}{< w_1, w_1 >} w_1 - \frac{< v_3, w_2 >}{< w_2, w_2 >} w_2 =$$

$$(1,4,2,8) - \frac{15}{15}(1,-2,-1,3) - \frac{40}{40}(4,4,2,2) =$$

$$(-4,2,1,3).$$

Setting $u_i = \frac{w_i}{||w_i||}$ for each i gives an orthonormal basis $\{u_1, u_2, u_3\}$ where

$$u_1 = (\frac{\sqrt{15}}{15}, \frac{-2\sqrt{15}}{15}, \frac{-\sqrt{15}}{15}, \frac{\sqrt{15}}{5})$$

$$u_2 = (\frac{\sqrt{10}}{5}, \frac{\sqrt{10}}{5}, \frac{\sqrt{10}}{10}, \frac{\sqrt{10}}{10})$$

$$u_3 = (\frac{-2\sqrt{30}}{15}, \frac{\sqrt{30}}{15}, \frac{\sqrt{30}}{30}, \frac{\sqrt{30}}{10}).$$

Finally, if $x = (-1, 2, 1, 1)$ the Fourier coefficients of x with respect to β are $< x, u_1 >= \frac{-\sqrt{15}}{5}, < x, u_2 >= \frac{2\sqrt{10}}{5}, < x, u_3 >= \frac{2\sqrt{30}}{5}$.

Section 6.3 #2a: $y = (1, -2, 4)$.

Section 6.3 #2b: $y = (1, -2)$.

Section 6.3 #3a: Let $T^*(x) = (r, s)$. Then

$$< T(1,0), x >=< (2,1), (3,5) >= 11 =$$

$$< (1,0), T^*(x) >=< (1,0), (r,s) >= r$$

and

$$< T(0,1), x >=< (1,-3), (3,5) >= -12 =$$

$$< (0,1), T^*(x) >= s.$$

Therefore $T^*(x) = (11, -12)$.

Section 6.3 #3b: Let $T^*(x) = (r, s)$. Then

$$< T(1,0), x >=< (2, 1-i), (3-i, 1+2i) >= 2(3+i) + (1-i)(1-2i) = 5-i =$$

$$< (1,0), T^*(x) >=< (1,0), (r,s) >= \bar{r}$$

and

$$< T(0,1), x >=< (i,0), (3-i, 1+2i) >= i(3+i) = -1+3i$$

$$< (0,1), T^*(x) >= \bar{s}.$$

Therefore $T * (x) = (5+i, -1-3i)$.