Calculus — Math 152 (Sections 21,22,23)
Exam 2 - Solutions to White exam
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1. A block of metal is heated to $300°$ F and placed in a container of liquid at $60°$ F. Let $y(t)$ denote the temperature of the block of metal at time t.

   (a) Assuming that Newton's Law of cooling applies to this situation, write a differential equation for $y(t)$. Identify the unspecified constant in the equation.

   Solution:
$$y' = -k(y - 60).$$
   The unspecified constant k is the cooling constant.

   (b) After one minute has elapsed, the block of metal is at $200°$ F. Use this information to determine the unspecified constant in the equation. (Since you don't have a calculator, you should leave your answer as an expression which may involve sum, difference, product, quotient, exponentiation and logarithms and no variables.)

   Solution: The general solution to the above equation is given by:
$$y = 60 + Ce^{-kt}.$$
   Here C is another constant that depends on the initial conditions. We are given that $y(0) = 300$, and substituting into the equation above gives $300 = 60 + C$, so $C = 240$. Thus:
$$y = 60 + 240e^{-kt}.$$
   We are also given that when $t = 1$ (in minutes) then $y = 200$. Therefore
$$200 = 60 + 240e^{-k}.$$
   Solving for k gives:
   $e^k = 240/140$ and so $k = \ln(240/140)$.

2. (a) Find all points of intersection of the curves $r = 2 + (\sin(\theta))^2$ and $r = 2 + (\cos(\theta))^2$ that have positive y coordinate. Express your answers in *rectangular coordinate*. (Your answers should not contain any trig functions or inverse trig functions in them.)

   Solution: Notice that the r values in both curves are always positive, so at any point of intersection the r values on the two curves must be the same.
   Thus at a point of intersection we have $2 + (\sin(\theta))^2 = 2 + (\cos(\theta))^2$ so $\sin(\theta)^2 = \cos(\theta)^2$. Since $\cos(\theta)^2 = 1 - \sin(\theta)^2$ we conclude that $2\sin(\theta)^2 = 1$ and $\sin(\theta)^2 = \cos(\theta)^2 = 1/2$.
   Therefore at a point of intersection we have:
$$r = 2 + 1/2 = \frac{5}{2}.$$
   Also:
   $\sin(\theta) = 1/\sqrt{(2)}$ or $\sin(\theta) = -1/\sqrt{(2)}$
   and

$\cos(\theta) = 1/\sqrt(2)$ or $\cos(\theta) = -1/\sqrt(2)$.
The rectangular coordinates of the points of intersection have the form $(r\sin(\theta), r\cos(\theta))$ and so they must be:
$(\pm\frac{5}{2\sqrt{2}}, \pm\frac{5}{2\sqrt{2}})$.
The problem asks only for points with positive y coordinate so the desired points of intersection are:
$(\frac{5}{2\sqrt{2}}, \frac{5}{2\sqrt{2}})$ and $(\frac{5}{2\sqrt{2}}, -\frac{5}{2\sqrt{2}})$.

(b) Write a definite integral representing the area of the region lying in the first quadrant that is bounded by the y-axis, the line $x = y$ and the curve $r = 2 + (\sin(\theta))^2$. (You are not asked to evaluate the integral.)

Solution: Sweeping out counterclockwise, the region starts at $x = y$ which is the angle $\theta = \pi/4$ and ends at the y axis, which is the angle $\theta = \pi/2$. (Draw a picture!) Using the formula for the area in polar coordinates we get

$$\int_{\pi/4}^{\pi/2} r^2 d\theta = \int_{\pi/4}^{\pi/2} (2 + \sin(\theta)^2)^2 d\theta.$$

3. (**10 points**) Let $T_3(x)$ be the Taylor polynomial of degree 2 for the function $y = \sqrt{x}$ at $a = 4$.

(a) Find $T_2(x)$.

Solution:

$$\begin{array}{rr} y(x) = x^{1/2} & y(4) = 2 \\ y'(x) = 1/2x^{-1/2} & y'(4) = 1/4 \\ y''(x) = -1/4x^{-3/2} & y''(4) = -1/32 \end{array}$$

So,

$$T_2(x) = 2 + \frac{1}{4}x - \frac{1}{64}x^2.$$

(b) Find an upper bound on the error if we use $T_2(4.5)$ to approximate $\sqrt{4.5}$. (Your answer should be an expression involving sum, difference, product and quotients of rational numbers but should not involve exponentiating to fractional powers.)

Solution: The error bound says that $|f(4.5) - T_2(4.5)| \leq \frac{K(4.5-4)^3}{3!} = \frac{K}{2^3(3!)}$, where K is an upper bound on $|y'''(x)|$ for $x \in [4, 4.5]$. So we need to determine such an upper bound. We calculate $|y'''(x)| = 3/8x^{-5/2}$. By inspection $|y'''(x)|$ is a decreasing function of x, therefore for $x \in [4, 4.5]$, the largest it can be is $|y'''(4)| = 3/2^8$. Overall, the upper bound on error is:

$$\frac{3}{2^3(3!)2^8} = 2^{-12}$$

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4. For each improper integral below, determine whether it converges or diverges (and explain your answer). Evaluate those that converge.

(a) $\int_1^\infty \frac{dx}{x+4x^2}$ (Hint: partial fractions)

Solution:

$$\frac{1}{x+4x^2} = \frac{1}{(1+4x)x}.$$

We look for constants A and B so that

$$\frac{1}{(1+4x)x} = \frac{A}{1+4x} + \frac{B}{x} = \frac{Ax + B(1+4x)}{(1+4x)x}.$$

This leads to $1 = B + x(A + 4B)$ which implies $B = 1$ and $A = -4$. So our integral is equal to:

$$\begin{aligned}
\int_1^\infty \frac{dx}{x} - \frac{4dx}{1+4x} &= \lim_{R\to\infty} \int_1^R \frac{dx}{x} - \frac{4dx}{1+4x} \\
&= \lim_{R\to\infty} \ln(x) - \ln(1+4x)|_{x=1}^{x=R} \\
&= \lim_{R\to\infty} \ln(R) - \ln(1+4R) - (\ln(1) - \ln(5)) \\
&= \lim_{R\to\infty} \ln(\frac{R}{1+4R}) + \ln(5)) \\
&= \lim_{R\to\infty} \ln(\frac{1}{4+1/R}) + \ln(5)) \\
&= \ln(5) - \ln(4).
\end{aligned}$$

(b) $\int_0^1 \frac{dx}{x+4x^2}$

Solution: We can reuse the work from the previous part:

$$\begin{aligned}
\int_0^1 \frac{dx}{x} - \frac{4dx}{1+4x} &= \lim_{R\to 0} \int_R^1 \frac{dx}{x} - \frac{4dx}{1+4x} \\
&= \lim_{R\to 0} \ln(x) - \ln(1+4x)|_{x=R}^{x=1} \\
&= \lim_{R\to 0} \ln(1) - \ln(5) - (\ln(R) - \ln(4R+1)) \\
&= \lim_{R\to 0} \ln(\frac{4R+1}{5R}) \\
&= \lim_{R\to 0} \ln(\frac{4+1/R}{5}).
\end{aligned}$$

As R goes to 0, this blows up to ∞, so the integral diverges.

5. Consider the equation given parametrically by $x(t) = \sqrt{t+1} + t$ and $y(t) = 1 + 1/t$ for $t > 0$.

(a) Show that the points $(1+\sqrt{2}, 2)$ and $(5, 4/3)$ lie on this curve.

Solution: When $t = 1$ we have $(x(t), y(t)) = (\sqrt{(}2) + 1, 2)$ and when $t = 3$ we have $(x(t), y(t)) = (5, 4/3)$.

(b) Write down an integral that represents the length of the arc along the curve given by the above equations between the points $(1+\sqrt{2}, 2)$ and $(5, 4/3)$. (You need not evaluate the integral.)

Solution: By part (a), the value of t in the parametric curve ranges from $t = 1$ to $t = 3$. Using the formula for length of a parametric curve we get:

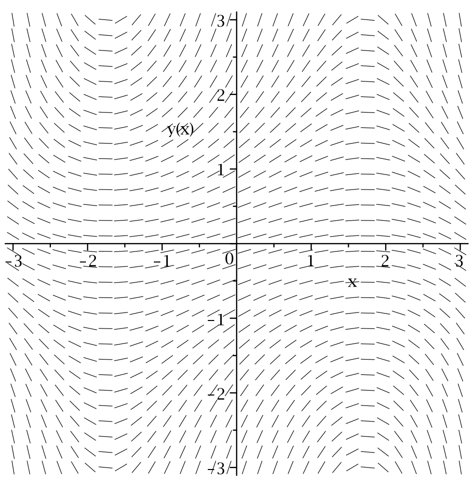
$$\int_{t=1}^{t=3} \sqrt{(x'(t))^2 + (y'(t))^2}dt = \int_{t=1}^{t=3} \sqrt{((t+1)^{-1/2}/2 + 1)^2 + (-1/t^2)^2}dt.$$

(c) Determine the equation of the line tangent to the curve at the point $(5, 4/3)$

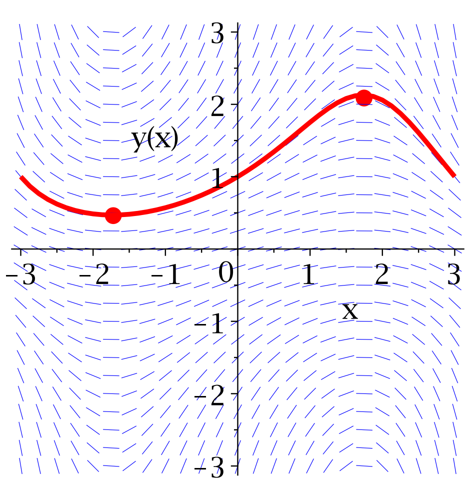
Solution: Using the point-slope form of the line, the equation is $y - 4/3 = m(x - 5)$ where m is the slope of the line. The slope is equal to $\frac{dy}{dx} = y'(t)/x'(t)$ at the given point. From part (a), the value of t at this point is $t = 3$. Calculating $y'(3) = -1/9$ and $x'(3) = 1 + 1/4 = 5/4$ so the slope is $-4/45$ and the equation of the line is:

$$y - 4/3 = -\frac{4}{45}(x - 5).$$

6. (**12 points**) The slope field for the differential equation $\frac{dy}{dx} = \frac{(3-x^2)(1+y^2)}{10}$ over the range $x, y \in [-3, 3]$ is shown below.



(a) In the figure above, or in a separate figure sketch the solution to the corresponding initial value problem with $y(0) = 1$ over the interval $[-3, 3]$, and indicate any critical points.



By looking at the original differential equation we can determine the x values of the two critical points exactly. At the critical point $\frac{dy}{dx} = 0$ and so $(3 - x^2)(1 + y^2) = 0$. Since $1 + y^2 > 0$ we must have $3 - x^2 = 0$. Therefore the x values of the critical points are $\sqrt{3}$ and $-\sqrt{3}$.

(b) Give an explicit formula of the form $y = f(x)$ for the solution to the above differential equation.

Solution Using separation of variables, we rewrite the differential equation as:

$$\frac{dy}{1+y^2} = \frac{1}{10}(3 - x^2)dx.$$

Integrating both sides gives:

$$\arctan(y) = [frac1103x - x^3/3 + C,$$

where C is an arbitrary constant. Thus:

$$y = \tan(\frac{1}{10}(3x - x^3/3) + C).$$

We can determine C from the fact that $y(0) = 1$:

$$1 = y(0) = \tan(C),$$

so we may take $C = \arctan(1) = \pi/4$.
Hence

$$y = \tan(\frac{1}{10}(3x - x^3/3) + \pi/4).$$

7. **(12 points)**

(a) Express the repeating decimal $0.383838\overline{38}$ as a fraction.

Solution: We can write this as a geometric series: $\sum_{n=0}^{\infty} .38 \times (.01)^n$ and using the formula for geometric series (which is valid here since $|.01| < 1$) we get

$$\frac{.38}{1 - .01} = \frac{.38}{.99} = \frac{38}{99}.$$

(b) Exactly evaluate the sum $\sum_{n=3}^{\infty} 1/(n^2 - 1)$. (Hint: Even though this is not an integral, the technique of partial fractions may help.)

Solution: Note that the denominator $n^2 - 1$ factors as $(n+1)(n-1)$ and so we look for A and B so that

$$\frac{1}{n^2 - 1} = \frac{A}{n-1} + \frac{B}{n+1} = \frac{A(n+1) + B(n-1)}{n^2 - 1}.$$

Equating the numerators $1 = A(n+1) + B(n-1)$ which implies $A - B = 1$ and $A + B = 0$ and so $A = 1/2$ and $B = -1/2$.
So we want to evaluate:

$$\sum_{n=3}^{\infty} \frac{1}{2(n-1)} - \frac{1}{2(n+1)}.$$

Writing out the first few terms:

$$(\frac{1}{4}-\frac{1}{8})+(\frac{1}{6}-\frac{1}{10})+(\frac{1}{8}-\frac{1}{12})+(\frac{1}{10}-\frac{1}{14})+\cdots,$$

we see this is a telecoping sum and that if we sum from $n=3$ to $n=T$ we get $\frac{1}{4}+\frac{1}{6}-\frac{1}{2T}-\frac{1}{2(T+1)}$.
Taking the limit as T goes to ∞ we get $\frac{1}{4}+\frac{1}{6}=\frac{5}{12}$.

8. (**15 points**) For each of the following series, determine whether it is divergent, conditionally convergent or absolutely convergent.

   In each case, justify your answer by clearly stating the tests used and how the test was applied.

   (a) $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n^2+n}}$

   Solution: This series converges but does not converge absolutely, so it converges conditionally.
   Let $a_n=(-1)^n/\sqrt{n^2+n}$.
   To see that the series converges we use the alternating sign test. The terms alternate in sign, and sequence $\{|a_n|\}$ is decreasing and converges to 0, so the alternating sign test implies that the series converges.
   To see that the series does not converge absolutely, we the limit comparison test. We compare $\sum_{n=1}^{\infty}|a_n|$ to $\sum_{n=1}^{\infty}1/n$. Now the ratio of $|a_n|/(1/n)=n/\sqrt{n^2+n}=1/\sqrt{1+1/n}$ which has limit 1 as n goes to ∞. Since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (by the integral test or the p-test with $p=1$), the limit comparison test implies that $\sum|a_n|$ also diverges.

   (b) $\sum_{n=1}^{\infty}(-1)^n(1-1/n)^n$

   This sequence diverges by the divergence test. Let $a_n=(-1)^n(1-1/n)^n$. We will show that a_n does not converge to 0, and so the divergence test implies the sum does not converge. To show that a_n does not converge to 0, we show that $\lim_{n\to\infty}|a_n|$ is equal to $1/e$. To do this we show that that $\lim\ln|a_n|=-1$. We have:

$$\begin{aligned}\ln|a_n| &= \ln(1-1/n)^n\\ &= n\ln(1-1/n).\\ &= \frac{\ln(1-1/n)}{1/n}.\end{aligned}$$

   As $n\to\infty$ both numerator and denominator tend to 0. So we use L'Hopital's rule to say that the limit of the ratio is the limit of the ratio of the derivatives:

$$\begin{aligned}\lim_{n\to\infty}\frac{\ln(1-1/n)}{1/n} &= \lim_{n\to\infty}\frac{(1/n^2)(1/(1-1/n))}{-1/n^2}\\ &= \lim_{n\to\infty}-1/(1-1/n)=-1.\end{aligned}$$

   To summarize, we have $\lim_{n\to\infty}\ln|a_n|=-1$, and therefore $\lim|a_n|=e^{-1}$. Thus the sequence $\{a_n\}$ bdoes not converge to 0, and so the series $\sum_{n=1}^{\infty}a_n$ diverges.

   (c) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n(\ln(n))^2}$

As discussed in class, this sum should have started at $n = 2$ since at $n = 1$ the first term is not defined.
Starting at $n = 2$ the sum converges absolutely. Let $c_n = (-1)^n/(n(\ln n)^2)$. We must show that $\sum_{n=2}^{\infty} |c_n|$ converges. Define the function f by $f(x) = 1/x(\ln(x))^2$. Then $|c_n| = f(n)$. Since f is a positive decreasing function of x we can use the integral test: Consider the integral $\int_2^{\infty} dx/(x\ln(x)^2) = \lim_{R\longrightarrow\infty} \int_2^R dx/(x\ln(x)^2)$. Using the substitution $u = \ln(x)$ this can be integrated to get

$$\begin{aligned} \lim_{R\to\infty}\int_2^{\infty} dx/(x\ln(x)^2) &= \lim_{R\to\infty} -1/\ln(x)|_{x=2}^{x=R} \\ &= \lim_{R\to\infty} 1/\ln(2) - 1/\ln(R) = 1/\ln(2). \end{aligned}$$

Since the integral converges we conclude that $\sum |c_n|$ converges and so $\sum c_n$ converges absolutely.