Extra problems on the pigeonhole principle (Version: September 22)

1. Let A be a subset of integers of size n. Prove that there is a subset of A whose sum is divisible by n.

2. Let A be a subset of size $n+1$ consisting of positive integers in the range 1 to $2n$. Prove that there must be distinct elements a, b of A such that a is a divisor of b

3. Let A be a finite subset of positive integers of size n Let $a_1, a_2, \ldots, a_t$ be a sequence of integers each belonging to A. Prove that if $t \geq 2^n$ then there are integers j, k satisfying $1 \leq j \leq k \leq n$ such that $\prod_{i=j}^{k} a_i$ is a perfect square.

4. Let M be a matrix of real numbers, with each row in nondecreasing order. Suppose we sort each column into nondecreasing order. Prove that the rows are still in nondecreasing order.

5. (Somewhat difficult) Let A, B be integer 2 by 2 matrices. Suppose that each of the matrices $A, A+B, A+2B, A+3B, A+4B$ has the property that it is invertible and its inverse has integer entries. Prove that $A+5B$ has the same property.