MATH 640:491–Fall, 2007
Problems to be discussed 10/23/07

1. Book: 4.1.8, 4.1.9, 4.2.8, 4.2.13, 4.3.21.

2. If the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has distinct zeros $r_1, \ldots, r_n$, all of which are non-zero, prove that

$$\frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_3} = -\frac{a_1}{a_0}.$$

3. Consider the polynomials $\binom{x}{0} = 1$, $\binom{x}{1} = x$, $\binom{x}{2} = \frac{x(x-1)}{2}$, $\binom{x}{3} = \frac{x(x-1)(x-2)}{3!}$, ...

   (a) Prove that any polynomial $P(x) \in \mathbb{C}[x]$ of degree n can be written uniquely as

$$P(x) = a_0\binom{x}{0} + a_1\binom{x}{1} + \cdots + a_n\binom{x}{n}$$

   with $a_i \in \mathbb{C}$.

   (b) Prove that $P(x)$ has integer coefficients if and only if $a_i \in \mathbb{Z}$ for each i.

4. Let $P(x) \in \mathbb{C}[x]$ be a polynomial. Assume that $P(z) \in \mathbb{R}$ if and only if $z \in \mathbb{R}$. Show that $P(x)$ has real coefficients and degree 1.

5. Let $P(x), Q(x) \in \mathbb{C}[x]$ be polynomials such that $P(x)^2 - Q(x)^3 = 1$. Prove that $P(x)$ and $Q(x)$ are both constant.

6. If $ax^2 + bx + c$ and $px^2 + qx + r$ have a common zero, then $(ar - cp)^2 = (aq - bp)(br - cq)$.

7. (USA Olympiad 1977) If $a \neq b$ are roots of $x^4 + x^3 - 1 = 0$, then ab is a root of $x^6 + x^4 + x^3 - x^2 - 1 = 0$.

8. (Pentagon 1987, modified) Let $a, b, c \in \mathbb{R}$ and assume that $b^2 < 3ac$. Show that the equation $x^3 + ax^2 + bx + c = 0$ has one real solution and two non-real solutions.