Problems

1. How many 0's does 400! end with?

2. Find the smallest positive integer having exactly 100 positive divisors.

3. Define $a_1 = 1$, and for every $n > 1$, define $a_{n+1} = a_n + \frac{1}{a_n}$. Prove that $20 < a_{200} < 24$.

4. If S is a set of real numbers let $S + S$ be the set of all sums of the form $a + b$ where $a \in S$ and $b \in S$ (where a, b are allowed to be the same number). For each positive integer n, how should you choose S of size n if you want to minimize the size of $S + S$?

5. Among all lists of positive integers that sum to 1000, what is the most their product can be?

6. Find all integers a, b such that $a^b = b^a$.

7. Determine all polynomials $P(x)$ that satisfy the equations $P(x^2+1) = (P(x))^2+1$ and $P(0) = 0$.

8. Let us say that a positive real number x is a *Fermat number* if there are three distinct positive integers a, b, c such that $a^x + b^x = c^x$. Prove that there exist arbitrarily large Fermat numbers (which means that for every real number M there is a fermat number x that is bigger than M).

9. Consider the following process which involves selecting a sequence $a_1, a_2, \ldots$ of numbers in the interval $(0, 1)$. Each number chosen earns a certain score as follows. The first number a_1 divides the interval $(0, 1)$ into the intervals $(0, a_1)$ and $(a_1, 1)$. The score earned is the product of the lengths of the two newly created intervals times the length of the original interval which is $a_1 \times (1 - a_1) \times 1$. Now after you've selected $n - 1$ points, you've divided $(0, 1)$ into n intervals. The next selected point a_n lands in one of those intervals and splits it. The score earned for choosing a_n is again the product of the lengths of the two newly created interval times the length of the interval that was just split.

   The total score is the sum of the scores. How should you choose the infinite sequence $a_1, a_2, \ldots$ so as to maximize the score. What is the maximum score possible?

10. Suppose we define the *linear size* of a box B, denote $LS(B)$ to be the sum of its length, width and height. Is it possible to construct two boxes B, C where $LS(B) < LS(C)$ and C fits inside of B?

11. Given 6 points in the plane, show that the ratio of the maximum distance between two of them to the minimum distance between two of them is at least $\sqrt{3}$.

12. Suppose we color the points of the x–y plane with three colors. Prove that there must be two points at distance one from each other that get the same color.

13. Evaluate $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}}$. (More formally, let $x_1 = \sqrt{2}$, and let $x_{n+1} = \sqrt{2}^{x_n}$ for each $n \geq 1$. Find $\lim_{n\to\infty} x_n$, if it exists.)

14. Standard U.S. coins are pennies (1 cent), nickels (5 cents), dimes (10 cents), quarters (25 cents), half dollars (50 cents) and dollars (100 cents). Say that an integer k is *convertible* provided that it is possible to find a collection of k coins that adds up to a dollar. What is the smallest positive integer that is not convertible?

15. Evaluate the determinant of the $n \times n$ matrix whose (i, j)th entry is $a^{|i-j|}$.

16. For a nonnegative integer n, let $f(n)$ be the number of ways to express n as a sum of powers of 2, where each power of 2 is used at most twice. For example, $f(6) = 3$ since $6 = 4 + 2 = 4 + 1 + 1 = 2 + 2 + 1 + 1$. (We define $f(0) = 1$). For a positive integer n. let $r(n) = f(n)/f(n-1)$. The function r is a rather amazing function: it is a bijection from the set of positive integers to the set of the positive rational numbers! Prove this.