The Pigeon-Hole Principle

1. Let $A$ be a subset of integers of size $n$. Prove that there is a nonempty subset of $A$ whose sum is divisible by $n$.

2. Given 19 distinct integers from the arithmetic progression $1, 4, 7, \ldots, 100$, prove that there are two entries that sum to 104.

3. Let $q$ be an odd integer greater than 1. Show that there is an integer $n$ such that $q$ divides $2^n - 1$.

4. Let $A = \{1, 2, 3, \ldots, 100\}$. Let $B \subseteq A$ be such that for all distinct $x, y \in B$, $x + y$ is not divisible by 11. Show that $|B| \leq 47$.

5. Let $A$ be a subset of size $n + 1$ consisting of positive integers in the range 1 to $2n$. Prove that there must be distinct elements $a, b$ of $A$ such that $a$ is a divisor of $b$.

6. If $S$ is a set of $2^n + 1$ points $\mathbb{R}^n$ with integer coordinates, then there are two points in $S$ such that the midpoint of the segment between them has all integer coordinates.

7. Suppose we have 25 points inside a regular hexagon of side-length 2. Show that some two of them are within distance 1 of each other.

8. Let $X$ be a real number and $n$ a positive integer. Prove that at least one of the numbers $X, 2X, \ldots, nX$ is within $1/((n + 1)$ of an integer.

9. Let $A$ be a finite subset of positive integers of size $n$. Let $a_1, a_2, \ldots, a_t$ be a sequence of integers each belonging to $A$. Prove that if $t \geq 2^n$ then there are integers $j, k$ satisfying $1 \leq j \leq k \leq n$ such that $\prod_{i=j}^{k} a_i$ is a perfect square.

10. Suppose $S$ is a subset of $\{1, 2, \ldots, 2n + 1\}$ such that for any two distinct elements $a, b \in S$, their sum $a + b$ is not in $S$. Show that $|S| \leq n + 1$.

11. Let $M$ be a matrix of real numbers, with each row in nondecreasing order. Suppose we sort each column into nondecreasing order. Prove that the rows are still in nondecreasing order.

12. Suppose 6 circles have a point in common. Prove that one of the circles contains the center of another circle.

13. Let $B$ be a subset of $\{-1, 1\}^n$ (the set of points in $\mathbb{R}^n$ with coordinates $-1$ or $+1$). If $|B| > 2^{n+1}/n$, prove that $B$ contains a set of three points that are the vertices of an equilateral triangle.

14. Let $m, n$ be positive integers. Suppose $x_1, \ldots, x_m$ are positive integers between 1 and $n$ and $y_1, \ldots, y_n$ are positive integers between 1 and $m$. Prove that there is a nonempty consecutive subsequence of $x_1, \ldots, x_m$ and a nonempty consecutive subsequence of $y_1, \ldots, y_n$ that have the same sum.

15. (Somewhat difficult) Let $A, B$ be integer 2 by 2 matrices. Suppose that each of the matrices $A, A + B, A + 2B, A + 3B, A + 4B$ has the property that it is invertible and its inverse has integer entries. Prove that $A + 5B$ has the same property.