Problems in elementary number theory

1. For any integer $n \geq 0$, the nth Fibonacci number is defined via the following recurrence: $F_n = F_{n-1} + F_{n-2}$, where $F_0 = 1$ and $F_1 = 1$. Prove that any two successive Fibonacci numbers are relatively prime. (Two integers are relatively prime if they have no common factor other than 1).

2. Show that the ten's digit of any power of 3 is even.

3. Prove that $b_n = \sum_{i=0}^{n} 10^i$ is a perfect square if and only if $n = 0$.

4. Determine all integers n such that $n^4 - n^2 + 64$ is a perfect square.

5. Show that $1^{1989} + 2^{1989} + \ldots + 1990^{1989}$ is a multiple of 1991.

6. Show that if $2n + 1$ and $3n + 1$ are both perfect squares then n is divisible by 40.

7. Prove that every integer is a divisor of infinitely many Fibonacci numbers.

8. Find the largest 3-digit prime factor of $\binom{2000}{1000}$.

9. Let $f(X)$ be a polynomial with integer coefficients. Let $a_1, a_2, \ldots, a_n$ be integers such that $f(a_i) = a_{i+1}$ for $1 \leq i < n$, and $f(a_n) = a_1$. Prove that $n \leq 2$.

10. Consider triples (x_1, x_2, x_3) of positive numbers summing to 1. Call the triple *balanced* if all of the numbers are less than or equal to 1/2. Consider the following operation on a triple: double all of the entries and subtracting 1 from the largest entry. If we start from an unbalanced triple and repeatedly apply the operation, must we eventually get a balanced triple?

11. Show that the product of the side lengths of any right triangle with integer side lengths is divisible by 60.

12. Show that for every composite integer n there are positive integers x, y, z so that $xy + xz + yz + 1 = n$.

13. Say that a positive integer is *alternating* if when expressed in base 2, any two consecutive digits are different. Is the number of alternating primes finite or infinite?

14. For any positive natural number n, let $H(n) = 1 + 1/2 + 1/3 + \cdots + 1/n$. Prove that for $n \geq 2$, $H(n)$ is not an integer.

15. For an integer n, let $\phi(n)$ be the number of integers less than n that are relatively prime to n. Prove that for any integer n, the sum of $\phi(d)$ over all divisors d of n is equal to n.

16. Suppose x can be be written as a sum of two squares of integers, and y can be written as a sum of two squares of integers. Show that xy can also be written as a sum of two squares of integers.