Easy Putnam problems, 1985-2005

85A–1 Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that

(i) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and
(ii) $A_1 \cap A_2 \cap A_3 = \emptyset$.

Express your answer in the form $2^a 3^b 5^c 7^d$, where a, b, c, d are nonnegative integers.

86A–1 Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.

87A–1 Curves A, B, C and D are defined in the plane as follows:

$$\begin{aligned} A &= \left\{(x,y) : x^2 - y^2 = \frac{x}{x^2+y^2}\right\}, \\ B &= \left\{(x,y) : 2xy + \frac{y}{x^2+y^2} = 3\right\}, \\ C &= \left\{(x,y) : x^3 - 3xy^2 + 3y = 1\right\}, \\ D &= \left\{(x,y) : 3x^2y - 3x - y^3 = 0\right\}. \end{aligned}$$

Prove that $A \cap B = C \cap D$.

88A–1 Let R be the region consisting of the points (x, y) of the cartesian plane satisfying both $|x| - |y| \leq 1$ and $|y| \leq 1$. Sketch the region R and find its area.

89A–1 How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?

90A–1 Let

$$T_0 = 2, T_1 = 3, T_2 = 6,$$

and for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$

The first few terms are

$$2, 3, 6, 14, 40, 152, 784, 5168, 40576.$$

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.

91A–1 A 2×3 rectangle has vertices as $(0, 0)$, $(2, 0)$, $(0, 3)$, and $(2, 3)$. It rotates $90°$ clockwise about the point $(2, 0)$. It then rotates $90°$ clockwise about the point $(5, 0)$, then $90°$ clockwise about the point $(7, 0)$, and finally, $90°$ clockwise about the point $(10, 0)$. (The side originally on the x-axis is now back on the x-axis.) Find the area of the region above the x-axis and below the curve traced out by the point whose initial position is (1,1).

85B–1 Let k be the smallest positive integer for which there exist distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial

$$p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$

has exactly k nonzero coefficients. Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.

86B–1 Inscribe a rectangle of base b and height h in a circle of radius one, and inscribe an isosceles triangle in the region of the circle cut off by one base of the rectangle (with that side as the base of the triangle). For what value of h do the rectangle and triangle have the same area?

87B–1 Evaluate

$$\int_2^4 \frac{\sqrt{\ln(9-x)}\,dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}.$$

88B–1 A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \dots\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with x, y, z positive integers.

89B–1 A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $\dfrac{a\sqrt{b}+c}{d}$, where a, b, c, d are integers.

90B–1 Find all real-valued continuously differentiable functions f on the real line such that for all x,

$$(f(x))^2 = \int_0^x [(f(t))^2 + (f'(t))^2]\,dt + 1990.$$

91B–1 For each integer $n \geq 0$, let $S(n) = n - m^2$, where m is the greatest integer with $m^2 \leq n$. Define a sequence $(a_k)_{k=0}^{\infty}$ by $a_0 = A$ and $a_{k+1} = a_k + S(a_k)$ for $k \geq 0$. For what positive integers A is this sequence eventually constant?