Problems in Calculus and Analysis[1]

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that for every a, b, $\frac{f(a)+f(b)}{2} = f\left(\frac{a+b}{2}\right)$. Show that $f(x) = \alpha x + \beta$ for some α, β.

   If we drop the condition that f is continuous, does the same conclusion hold?

2. Suppose $f : \mathbb{R} \to R$ is a n-times differentiable function, and suppose $a_1 < a_2 < \ldots < a_{n+1}$ are such that $f(a_i) = 0$ for each $i \in \{1, \ldots, n+1\}$. Show that there is some $b \in [a_1, a_{n+1}]$ such that $f^{(n)}(b) = 0$.

3. Show that there is a unique real number c such that for every differentiable function $f : [0,1] \to \mathbb{R}$ with $f(0) = 0$ and $f(1) = 1$, the equation $f'(x) = cx$ has a solution.

4. Suppose $f(x)$ is differentiable and has at least two zeros on the interval $[a, b]$. Prove that for any constants c_0 and c_1, the function $c_0 f(x) + c_1 f'(x)$ has a root on the interval $[a, b]$.

5. Find all continuous functions f such that for every $x > 0$,

$$\int_1^x f(t)dt = \int_x^{x^2} f(t)dt.$$

6. Does there exist a collection $\mathcal{F}$ of uncountably many subsets of $\mathbb{N}$ such that for every $A, B \in \mathcal{F}$, either $A \subset B$ or $B \subset A$?

7. Let $f : [0,1] \longrightarrow (0,1)$ be continuous. Show that the equation $2x - \int_0^x f(t)dt = 1$ has exactly one solution on the interval $[0,1]$.

8. Suppose that n is a nonnegative integer and

$$f(x) = c_0 e^{r_0 x} + \cdots + c_n e^{r_n x}$$

   where $c_0, \ldots, c_n$ and $r_0, \ldots, r_n$ are real numbers with the r_i distinct. Prove that if f has more than n roots then $c_0 = c_1 = \cdots = c_n = 0$. (Hint: Use induction on n.)

9. Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ has the property that for all $a \in \mathbb{R}$, $\lim_{x \longrightarrow a} f(x)$ exists. Let S be the set points $b \in \mathbb{R}$ such that f is discontinuous at b. Prove that S is countable.

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