1. Suppose \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function such that for every \( a, b \), \[ \frac{f(a) + f(b)}{2} = f \left( \frac{a + b}{2} \right). \] Show that \( f(x) = \alpha x + \beta \) for some \( \alpha, \beta \).

If we drop the condition that \( f \) is continuous, does the same conclusion hold?

2. Suppose \( f : \mathbb{R} \to \mathbb{R} \) is a \( n \)-times differentiable function, and suppose \( a_1 < a_2 < \ldots < a_{n+1} \) are such that \( f(a_i) = 0 \) for each \( i \in \{1, \ldots, n+1\} \). Show that there is some \( b \in [a_1, a_{n+1}] \) such that \( f^{(n)}(b) = 0 \).

3. Show that there is a unique real number \( c \) such that for every differentiable function \( f : [0, 1] \to \mathbb{R} \) with \( f(0) = 0 \) and \( f(1) = 1 \), the equation \( f'(x) = cx \) has a solution.

4. Suppose \( f(x) \) is differentiable and has at least two zeros on the interval \( [a, b] \). Prove that for any constants \( c_0 \) and \( c_1 \), the function \( c_0 f(x) + c_1 f'(x) \) has a root on the interval \( [a, b] \).

5. Find all continuous functions \( f \) such that for every \( x > 0 \),
   \[ \int_1^x f(t) \, dt = \int_x^{x^2} f(t) \, dt. \]

6. Does there exist a collection \( \mathcal{F} \) of uncountably many subsets of \( \mathbb{N} \) such that for every \( A, B \in \mathcal{F} \), either \( A \subset B \) or \( B \subset A \)?

7. Let \( f : [0, 1] \to (0, 1) \) be continuous. Show that the equation \( 2x - \int_0^x f(t) \, dt = 1 \) has exactly one solution on the interval \( [0, 1] \).

8. Suppose that \( n \) is a nonnegative integer and
   \[ f(x) = c_0 e^{r_0 x} + \cdots + c_n e^{r_n x} \]
   where \( c_0, \ldots, c_n \) and \( r_0, \ldots, r_n \) are real numbers with the \( r_i \) distinct. Prove that if \( f \) has more than \( n \) roots then \( c_0 = c_1 = \cdots = c_n = 0 \). (Hint: Use induction on \( n \).)

9. Suppose \( f : \mathbb{R} \to \mathbb{R} \) has the property that for all \( a \in \mathbb{R} \), \( \lim_{x \to a} f(x) \) exists. Let \( S \) be the set points \( b \in \mathbb{R} \) such that \( f \) is discontinuous at \( b \). Prove that \( S \) is countable.