Problem Set 8.Matrices and Groups[1]

1. Let G be a set and $*$ be an associative binary operation. Suppose that for all $a, b \in G$ we have $a^2b = b = ba^2$. Show that G is a commutative group.
2. Supposethat G is a finite group and A is a subset of G of size more than $|G|/2$. Prove that for every $g \in G$ there are elements $a, b \in A$ such that $ab = g$.
3. Suppose G is a group, and $a, b \in G$ satisfy $ba = ab^{-1}$ and $ab = ba^{-1}$. Prove that $a^4 = b^4 = e$ (where e is the identity element.)
4. For a square matrix A, define $\sin A$ by the power series:

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$

Prove or disprove: there exists a 2×2 matrix A with real entries such that

$$\sin A = \begin{matrix} 1 & 1996 \\ 0 & 1 \end{matrix}.$$

5. Let G be a group with identity e and let $\phi : G \to G$ be a function such that $\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$ whenever

$$g_1g_2g_3 = e = h_1h_2h_3.$$

Prove that there exists an element $a \in G$ such that $\psi(x) = a\phi(x)$ is a homomorphism (i.e., for all x, y, we have $\psi(xy) = \psi(x)\psi(y)$).

6. If A and B are square matrices of the same size such that $ABAB = 0$, then must it be the case that $BABA = 0$?
7. Let S be a set of real numbers which is closed under multiplication. Let T and U be disjoint subsets of S whose union is S. Given that the product of any three elements of T is in T, and that the product of any three elements of U is in U, show that at least one of the sets T, U is closed under multiplication.
8. Let G be a finite set of real $n \times n$ matrices $M_1, ...M_r$ which form a group under matrix multiplication. Suppose that $\sum_{i=1}^{r} Tr(M_i) = 0$, where Tr denotes the trace (sum of the main diagonal). Prove that $\sum_{i=1}^{r} M_i$ is the $n \times n$ zero matrix.
9. Let G be a finite group of order n generated by a and b. Prove or disprove: there is a sequence $g_1, \ldots, g_{2n}$ such that every element of G occurs exactly twice, and g_{i+1} equals g_ia or g_ib for each $i = 1, \ldots, 2n-1$, and $g_1 = g_{2n}a$ or $g_{2n}b$.
10. Let G be a finite group with identity e. If g and h are two elements in G such that $g^3 = e$ and $ghg^{-1} = h^2$, then find the order of h.

[1] 11-3-15 version