Set 9: Problems on Geometry

1. If the angle A of a triangle ABC is doubled, but the lengths of the sides AB and AC are kept the same, the area of the triangle ABC stays the same. Find the angle A.
2. In a triangle ABC with side lengths a, b, c, angle A is twice angle B. Show that $a^2 = b(b + c)$.
3. Prove that if one angle of a triangle is equal to 120 degrees, then the triangle formed by the feet of the angle bisectors is right angled.
4. Show that any rectangle inscribed in an ellipse has its sides parallel to the axes of the ellipse (unless the ellipse is a circle).
5. Let C be a circle and x,y,z be three points on circle and T be the triangle formed by x,y,z. Let a_x be the length of the angle bisector at x, a_y be the length of the angle bisector at y and a_z be the length of the angle bisector at z. Let b_x be the length of the segment obtained by extending the angle bisector at x until it hits the circle on the other side of yz. Define b_y and b_z similarly. Prove that $\sqrt{a_x a_y a_z b_x b_y b_z}$ is equal to the product of the lengths of the sides of the triangle.
6. Determine the set of triples (a, b, c) such that the line $ax + by = 1$ us tangent to the circle $x^2 + y^2 = c^2$.
7. Let T_j be an equilateral triangle of sidelength $2j - 1$. For a fixed line L in the plane, place each triangle T_1,T_2,T_3, etc. so that one side of each triangle lies on L and each successive triangle shares a vertex with the previous. Let v_j be the vertex of T_j that is not on L. Prove that there is a parabola that passes through all of the points v_j and prove that all of the v_j have integerdistance from the focus of the parabola.
8. Let A,B,C be vertices of a triangle, and let L be the midpoint of side AB, M be the midpoint of side BC and N be the midpoint of side AC. Prove that there is another triangle whose side lengths are equal to $\bar{C}L$, $\bar{A}M$ and $\bar{B}N$.