MATH 642:582–Fall, 2003

Assignment 3–Due October 17 (Version: October 12)

1. Jukna, 2.11
2. Jukna, 2.12
3. Jukna, 2.17
4. Let n, j be a positive integers and $\alpha, \gamma \in (0,1)$. Let X be a set of size n and $\mathcal{A}$ be a set of subsets of X satisfying: (i) $|A| \geq \alpha n$ for all $A \in \mathcal{A}$ and (ii) The intersection of any j distinct sets of $\mathcal{A}$ has size at most $\gamma\alpha^j n$. Prove that the number of sets in $\mathcal{A}$ is at most $\frac{\binom{j}{2}}{\alpha^{j-1}(1-\gamma)}$.
5. A famous and long-standing conjecture of Caccetta and Häggkvist says that any n-vertex digraph with all out-degrees at least $n/3$ has a cyclically directed cycle of length 2 or 3. (It's amazing that this seems to be quite hard!)

   Say that β is *admissible* if there is a graph G satisfying condition $C(\beta)$: the out-degrees of G are at least $\beta|V(G)|$ and G has no directed cycle of length 2 or 3. The C-H conjecture is equivalent to saying that $1/3$ is not admissible.

   (a) Show that the result, if true, is tight: every real number less than $1/3$ is admissible.

   (b) The rest of this problem is devoted to proving a result of Shen: $(3-\sqrt{7})$ is not admissible. (This number is about .3542).

   Let β be an admissible number and let G be a minimal graph satisfying $C(\beta)$. For a vertex u write $N^+(u)$ (respectively, $N^-(u)$) for the set of out-neighbors (respectively, in-neighbors) of u. Prove: for every directed edge (u,v) of G, if $N^+(u) \cap N^+(v)$ is nonempty, then there is a vertex w of $N^+(u) \cap N^+(v)$ such that $|N^+(w) - N^+(v)| \geq (1-\beta)|N^+(u) \cap N^+(v)|$.

   (c) Prove that for any edge (u,v) of G, $n > |N^+(v)| + |N^-(u) \cup N^-(v)| + (1-\beta)|N^+(u) \cap N^+(v)|$.

   (d) Let t be the number of transitive triangles of G (i.e. triples a, b, c of distinct vertices with $(a,b), (b,c)$ and (a,c) all edges). Prove that $(3\beta - 1)n^3 \leq t$. (Hint: average the previous inequality over all edges of G).

   (e) Show by another argument that $t \leq \beta^2 n^3/2$ and deduce that $\beta \leq 3 - \sqrt{7}$.
6. Let M be a matrix with distinct real entries and suppose that each row of M is in increasing order from left to right. Construct a matrix M' by reordering each column of M so that the column is in increasing order from top to bottom. Prove that in M', the rows are still in increasing order from left to right.
7. A beautiful result in number theory says that every prime number that is congruent to 1 mod 4 can be expressed as the sum of two squares. There are various proofs of this in the literature. This problem develops one of the most elementary proofs.

   This problem develops ths steps needed for this proof (some of which may be familiar to you.) You may assume that every integer has a unique prime factorization, but may not use any facts from group theory or finite field theory. In particular, you may not use the fact that the non-zero integers mod p form a cyclic group under multiplication.

(a) Prove that for any prime p and any integer a that is not divisible by p there is a unique integer b between 1 and $p-1$ such that $ab-1$ is divisible by p. (Hint: for existence use proof by contradiction and the pigeonhole principle.)

(b) Prove that for any prime p, $(p-1)!+1$ is divisible by p. (Hint: use the previous part.)

(c) For p congruent to 1 mod 4, construct an integer x such that x^2+1 is divisible by p. (Hint: use the previous part.)

(d) Given any integer y and prime p show that there are integers j, k satisfying $1 \leq |j| < \sqrt{p}$ and $1 \leq k < \sqrt{p}$ such that $jy-k$ is divisible by p. (Hint: use the pigeonhole principle.)

(e) Prove the theorem.