A few preliminary remarks.

1. Follow the general instructions for homework given in:
   http://www.math.rutgers.edu/~saks/homework.html

2. Most of these problems are based on results that have appeared in research papers. Problems typically are broken into multiple steps that guide you through the solution.

3. For some problems hints are provided at the end of the assignment. This is noted by the statement “Hint given below.” I recommend you try to solve the problem without the hint and refer to the hint if you are stuck. If you use a hint you should acknowledge using it in your acknowledgements for the problem (as described in the general instructions for homework mentioned above.

4. Problems 3 and 4 involve some things that we will discuss in class and may be a bit easier after the discussion.

5. Please be on the look out for errors. If something seems not to make sense, check with me before investing a lot of time on the problem. I would appreciate being notified of any typos (even minor ones).

6. This homework uses the following definitions:
   - If $V$ is a set, a hypergraph $H$ on $V$ is a set of nonempty subsets of $V$. The members of $H$ are called edges and $V$ is called the vertex set.
   - If $H$ is a hypergraph on $V$, then $H^1$ is the hypergraph $\{W \subseteq V : \exists E \in H, W \subseteq E\}$ and $H^1$ is the hypergraph $\{W \subseteq V : \exists E \in H, E \subseteq W\}$
   - A matching or edge packing of $H$ is a subset of edges that are pairwise disjoint. The matching number $\nu(H)$ is the size of the largest matching of $H$.
   - A fractional matching is a nonnegative real-valued function $w$ with domain $H$ satisfying that for every vertex $v$, $\sum_{E:v \in E} w(E) \leq 1$. The fractional matching number $\nu^*(H)$ is the maximum of $\sum_{E \in H} w(E)$ over all fractional matchings of $H$.
   - A vertex cover (or blocking set of $H$ is a subset $C$ of vertices such that $C \cap E \neq \emptyset$ for all $E \in H$. $H^B$ is the set of blocking sets of $H$. $\tau(H)$ is the minimum size of a vertex cover.
   - A fractional vertex cover of $H$ is a nonnegative real valued function $\kappa$ defined on $V$ with the property that for every edge $E$, $\sum_{v \in E} w(v) \geq 1$. The fractional cover number $\tau^*(H)$ is the minimum of $\sum_v w(v)$ over all fractional vertex covers.
   - The incidence matrix $M = M(H)$ of $H$ is the matrix with rows indexed by edges and columns indexed by vertices, where $M_{E,v} = 1$ if $v \in E$ and is 0 otherwise.
   - A hypergraph is intersecting if $E \cap F \neq \emptyset$ for all $E, F \in H$ and is $k$-wise intersecting if $E_1 \cap \cdots \cap E_k \neq \emptyset$ for all $E_1, \ldots, E_k \in H$.
   - For a graph $G = (V, E)$, a clique is a subset of vertices that are pairwise adjacent, and an independent set is a subset of vertices that are pairwise non-adjacent.
1. (a) Recall that a sequence of disjoint set pairs (SDSP) is a sequence \((A_1, B_1), \ldots, (A_t, B_t)\) such that \(A_i \cap B_i = \emptyset\). An SDSP is weakly crossing if for each \(i \neq j\), \(A_i \cap B_j \neq \emptyset\) or \(A_j \cap B_i \neq \emptyset\). Prove that for a weakly crossing SDSP, \(\sum_{i=1}^{t} 2^{-|A_i|+|B_i|} \leq 1\).

(Hint given below.)

(b) The rest of this problem involves a nice application of the above theorem to combinatorial geometry. Some preliminaries:

- A set of points in \(\mathbb{R}^d\) is in **convex position** if no one of the points is in the convex hull of the others.
- A \(j\)-simplex is the convex hull of \(j+1\) distinct points in convex position.
- Warmup problem (not to be handed in): A \(d\)-simplex in \(\mathbb{R}^d\) is closed and compact. The boundary of a \(d\)-simplex is the union of \(d+1\) \((d-1)\)-simplices (called the facets of the simplex) having disjoint interiors. Each facet of the simplex lies in a unique \(d-1\)-dimensional hyperplane, which is called a **bounding hyperplane** of the simplex.
- Warmup problem (not to be handed in): Two \(d\)-simplices are said to be **adjacent** if there is a unique \(d-1\) dimensional hyperplane containing their intersection. Show that this hyperplane must be a bounding hyperplane of both simplexes.

Prove that a collection of \(d\)-simplices in \(\mathbb{R}^d\) that are pairwise adjacent has size at most \(2^{d+1}\).

2. Recall that a hypergraph \(\mathcal{H}\) is said to **shatter** a set \(S\) if the trace \((\mathcal{H}|S)\) (i.e., \(\{E \cap S : E \in \mathcal{H}\}\)) is equal to \(2^S\). The VC-dimension of \(\mathcal{H}\) is the size of the largest set shattered by \(\mathcal{H}\). A well known theorem proved independently by Shelah and Sauer says that any \(n\) vertex hypergraph with VC-dimension \(\leq d\) has at most \(\sum_{i=0}^{d} \binom{n}{i}\) sets. (This bound is tight since the hypergraph \((\binom{n}{d})\) consisting of all subsets of size at most \(d\) has VC-dimension \(d\).) There are various proofs known for this, here we’ll use the linear algebraic method.

(a) Consider the space \(\mathbb{R}[x_1, \ldots, x_n]\) of polynomials. For \(S \subseteq [n]\) let \(x^S = \prod_{i \in S} x_i\). Let \(M_n\) denote the polynomial space spanned by \(\{x^S : S \subseteq [n]\}\). For \(S\) of \([n]\), define the polynomial \(p_S(x_1, \ldots, x_n) = \prod_{i \in S}(1 + x_i)\prod_{i \notin S}(1 - x_i)\). Show that the collection of polynomials \(\{p_S : S \subseteq [n]\}\) is a basis for \(M_n\).

(b) Recall that if \(a \in \mathbb{R}^n\), the **evaluation functional** \(E_a\) maps \(M_n\) to \(\mathbb{R}\) by mapping the polynomial \(p\) to \(p(a)\). Suppose that for each subset \(B\) of \([n]\), \(a^B \in \mathbb{R}^n\) is an arbitrary point with **support** \(B\) (which means that \(a^B_i \neq 0\) if and only if \(i \in B\).) Prove that the set of evaluation functionals \(\{E_{a^B} : B \subseteq [n]\}\) is linearly independent. (Hint given below).

(c) Now let \(\mathcal{H}\) be a hypergraph with VC-dimension at most \(d\) and consider the collection \(\{p_S : S \in \mathcal{H}\}\). Show that for each \(B \subseteq [n]\) of size bigger than \(d\) there is a point \(a^B\) with support \(B\) with the property that \(p_S(a_B) = 0\) for all \(S \in \mathcal{H}\).

(d) Complete the proof of the theorem.

3. A hypergraph \(\mathcal{H}\) is **\(\tau\)-critical** if for any edge \(E\) of \(\mathcal{H}\), \(\tau(\mathcal{H} - \{E\}) < \tau(\mathcal{H})\). Let \(\mathcal{H}\) be a \(\tau\)-critical hypergraph. Prove that either the edges of \(\mathcal{H}\) are pairwise disjoint or \(\tau^*(\mathcal{H}) < \tau(\mathcal{H})\). (Hint given below)
4. Let $\mathcal{H}$ be a hypergraph on $V$ and $M = M(\mathcal{H})$ be its incidence matrix (defined above).

(a) Prove that if every square submatrix of $M$ has determinant in $\{0, 1, -1\}$ then $\nu(\mathcal{H}) = \tau(\mathcal{H})$.

(b) Prove that if $\mathcal{H}$ is a bipartite graph (so $V$ can be partitioned into two sets $V_1, V_2$ and every edge contains one vertex from $V_1$ and one from $V_2$) then every square submatrix of $M$ has determinant in $\{0, 1, -1\}$. (This provides another proof of König’s theorem for bipartite graphs).

5. König’s theorem says: for any bipartite graph $G$, $\tau(G) = \nu(G)$.

Dilworth’s theorem says: for any partially ordered set $P$, the width of $P$ (maximum size of an antichain) is equal to the minimum size of a cover of $P$ by chains.

(a) Deduce König’s Theorem from Dilworth’s theorem.

(b) Deduce Dilworth’s theorem from König’s Theorem. (Hint given below)

6. Let $A[J] = (A_j : j \in J)$ be an finite indexed family of subsets of the finite set $X$. A $J \times X$ matrix $M$ is compatible with $A[J]$ if for all $x \in X$ and $j \in J$, if $x \notin A_j$ then $M[j, x] = 0$.

(a) Prove that $A[J]$ has a system of distinct representatives (i.e. there is a injection $f : J \rightarrow X$ such that $f(j) \in A_j$ for all $j$) if and only if there is a nonnegative matrix compatible with $A[J]$ whose row sums are each at least 1, and whose column sums are each at most 1.

(b) Let $A[J]$ be as above. For $x \in X$, let $\text{deg}(x) = |\{j : x \in A_j\}|$. Prove: if for every $j \in J$ and $x \in A_j$, $|A_j| \geq \text{deg}(x)$, then $A[J]$ has a system of distinct representatives.

7. (a) Prove: if $\mathcal{H}$ is an $r$-uniform $r+1$-wise intersecting hypergraph, then $\tau(\mathcal{H}) = 1$. (A hypergraph is $r+1$-wise intersecting if any $r+1$ sets have nonempty intersection.)

(b) More generally, prove: if $\mathcal{H}$ is an $r$-uniform $k$-wise intersecting then $\tau(H) \leq \frac{r-1}{k-1} + 1$. (Hint given below.)

(c) For each $r, k \in \mathbb{N}$ such that $k - 1$ divides $r - 1$, give an example of an $r$-uniform $k$-wise intersecting hypergraph $\mathcal{H}(k, r)$ such that $\tau(\mathcal{H}) = \frac{r-1}{k-1} + 1$.

8. The purpose of this problem is to prove a generalization of the Hall’s marriage theorem. For an indexed family $A[\Gamma] = (A_\alpha : \alpha \in \Gamma)$ of sets, the surplus is $|\bigcup_{\alpha \in \Gamma} A_\alpha| - |\Gamma|$ (which might be positive or negative). The min-surplus of $A[\Gamma]$ is defined to be the minimum surplus of any indexed subfamily $A[\Gamma']$ where $\Gamma' \subseteq \Gamma$.

If $A(\Gamma)$ and $B(\Gamma)$ are indexed families of sets (with the same index set $\Gamma$) we say that $B$ is a pruning of $A$ if $B_\alpha \subseteq A_\alpha$ for all $\alpha \in \Gamma$.

**Theorem.** Suppose $A[\Gamma]$ has min-surplus $s \geq 0$. Then $A[\Gamma]$ has a pruning $B[\Gamma]$ with min-surplus $s$ and all of its sets of size exactly $s + 1$.

(Not to hand in: (i) Why is this a generalization of Hall’s theorem? (ii) In the case that $s = 1$ the conclusion is equivalent to: $A[\Gamma]$ can be pruned to a graph (all edges of size 2) that has no cycles, i.e., is a forest.)
(a) If $\Gamma' \subseteq \Gamma$ is nonempty and the surplus of $A[\Gamma']$ is equal to the min-surplus of $A[\Gamma]$, we say that $\Gamma'$ is surplus minimizing for $A[\Gamma]$. Prove that if $\Gamma_1$ and $\Gamma_2$ are surplus minimizing for $A[\Gamma]$ then $\Gamma_1 \cap \Gamma_2$ is either empty or is also surplus critical for $A[\Gamma]$.

(b) $\alpha \in \Gamma$ is unshrinkable for $A[\Gamma]$ if replacing $A_{\alpha}$ by any proper subset reduces the min-surplus of $A$. Prove that if $\alpha$ is unshrinkable for $A[\Gamma]$ then there is a surplus minimizing set $\Gamma'$ that contains $\alpha$, such that $A_{\beta} \cap A_{\alpha} = \emptyset$ for all $\beta \in \Gamma'$ with $\beta \neq \alpha$.

(c) Prove the theorem.

Hints:

**Problem 1** Consider random functions from the vertex set to $\{0, 1\}$.

**Problem 2** Part b. Consider the matrix with rows and columns indexed by subsets of $[n]$ with $A,B$ entry defined by $\prod_{i \in A} a_i^B$ and show that this matrix is nonsingular.

**Problem 3** For each edge $E$ of $\mathcal{H}$ there is a vertex cover of $\mathcal{H} - \{E\}$ of size less than $\tau(\mathcal{H})$. Use these covers to build a fractional cover of $\mathcal{H}$.

**Problem 5** Part b. Given a partially ordered set $P$, define a bipartite graph where the vertices on each side correspond to a copy of the set $P$, with the edges defined in some appropriate way.

**Problem 7** Part b. Prove by induction on $j$ the claim that for $j \leq k$, there is a set of $j$ edges whose intersection has size at most $r - (\tau(\mathcal{H}) - 1)(j - 1)$. 

4