1. Consider the multi-commodity flow problem: Given a graph $G = (V, E)$ with a non-negative valued capacity function $C$ on the edges and $k$ triples $(s_i, t_i, D_i)$, $1 \leq i \leq k$ where $s_i, t_i \in V$ and $D_i$ is a nonnegative real number. A flow is a set of $k$ functions $f_1, \ldots, f_k$ defined on $V \times V$ such that: (1) for every $i \in [k]$ and each $v \in V - \{s_i, t_i\}$, $\sum_w f_i(w, v) = \sum_w f_i(v, w)$. (2) For each edge $e = vw$, $\sum_i f_i(v, w) + f_i(w, v) \leq C(e)$. The value of the flow is the vector $(V_1, \ldots, V_k)$ where $V_i = \sum_w f_i(w, t_i)$. The maximum concurrent flow problem is to find a flow that maximizes $\min_i \{V_i / D_i\}$.

(a) Let $P_i$ denote the set of paths from $s_i$ to $t_i$. Formulate the maximum concurrent flow problem as a linear program whose variables correspond to elements of $\bigcup_i P_i$.

(b) Form the dual of the LP of the first part.

(c) Show that the optimal solution of the dual is equal to the minimum over all metrics $\rho$ defined on $G$ of the ratio $\sum_{vw \in E} C(v, w) \rho(v, w) / \sum_{i=1}^k D_i \rho(s_i, t_i)$.

2. Let $M = (X, \rho)$ be a finite metric space on $n$ vertices. Recall that the tree volume $TVOL(M)$ is the minimum over all trees on $X$ of the product of the edge lengths, and the harmonic volume $HVOL(M)$ is the harmonic mean over all $n!$ orderings $x_1, \ldots, x_n$ of the points of $\prod_{i=1}^{n-1} \rho(x_i, x_{i+1})$. Prove that $nTVOL(M) \geq 2^{n-1} HVOL(M)$. (Hint: One way to do this is by induction on $n$).

3. The purpose of this problem is to prove a result stated in class that every metric space has a probabilistic embedding into the real line whose probabilistic distortion is $O(\log n)$. Here's the set up.

- Let $(X, \rho)$ be a metric space on $n$ points.
- Let $b \geq 1$ and $S$ be a positive integer.
- For $k \geq 0$, let $p_k = \frac{1}{2^b k}$.
- Let $P$ be a random variable on $[0, 1]$ obtained by selecting uniformly from the set $\{p_0, p_1, \ldots, p_{S-1}\}$.
- Let $A$ be a random subset of $X$ obtained as follows. Select $P$ as above. Then for each $x \in X$, put $x$ in $A$ independently with probability $P$.
- Define $f : X \rightarrow \mathbb{R}$ to be $f(x) = \rho(x, A)$. (So $f$ is a random mapping depending on $A$).

As we noted in class, this is non-expansive (independent of $A$). We will show:

**Claim.** Assume $b^S \geq n$. For each pair $u \neq v \in X$, $\mathbb{E}[|f(u) - f(v)|] \geq \Omega(\rho(u, v)/Sb)$. (Note that when $b = 2$ and $S = \log n$ we get the result stated at the beginning of the problem.)

Fix $u, v \in X$ Here is an outline of a proof of this claim; you’ll be asked to provide some details.
(a) For a point \( x \in X \), let \( g(x) = \rho(x, \{u, v\}) \) and let \( Y = \{ x : g(x) \leq \rho(u, v)/2 \} \). Order the points of \( Y \) as \( y_1, \ldots, y_s \) in nondecreasing order of \( g(x) \). For \( 1 \leq i \leq s \), let \( g_i = g(y_i) \) and let define \( g_{s+1} = \rho(u, v)/2 \). Define \( Z_i \) for \( 1 \leq i \leq s \) to be the random variable that is 1 if \( \min(f(u), f(v)) \leq g_i \leq g_{i+1} \leq \max(f(u), f(v)) \). Prove that \( |f(u) - f(v)| \geq \sum_{i=1}^{n} Z_i(g_{i+1} - g_i) \).

(b) Prove that for each \( i \in [s] \) \( \Pr[Z_i = 1] = \Omega(1/Sb) \). (This is the main step. If you need an additional hint, ask me.)

(c) Prove the claim.

4. Let \( T_n \) be the balanced rooted binary tree of depth \( n \) (which has \( 2^{n+1} - 1 \) vertices) with the usual graph distance metric \( d_T \). The depth of a vertex is its distance from the root. View the vertex set \( V = V_n \) of \( T \) as the set of binary strings of length \( \leq n \) (where the root is the empty string \( \Lambda \) and the children of \( \alpha \) are \( \alpha 0 \) and \( \alpha 1 \)). Write \( \alpha \prec \beta \) if \( \alpha \) is a prefix of \( \beta \) which means that it is an ancestor of \( \beta \) in the tree. If \( \beta \in \{0, 1\}^n \), let \( \beta(j) \) be the prefix of \( \beta \) of length \( j \), which corresponds to the \( j \)th node on the path from the root to the leaf \( \beta \).

The purpose of this problem is to give a relatively simple proof of a theorem of Bourgain: every embedding of a depth \( n \) binary tree into \( \ell_2 \) has distortion \( \Omega(\sqrt{\log n}) \). Throughout \( \| \cdot \| \) denotes the Euclidean norm.

(a) Prove that for any three vectors \( a, b, c \in \mathbb{R}^n \) we have \( \|a - 2b + c\|^2 = 2\|a - b\|^2 + 2\|b - c\|^2 - \|a - c\|^2 \).

(b) Let \( A = A(n) = \{(p, j) : 2^j \leq \min\{p, n-p\}\} \). Given a sequence \( \bar{x} = x_0, x_1, \ldots, x_n \) where each \( x_i \) is a vector in \( \ell_2 \) we define \( \lambda(\bar{x}, p, j) = \|x_{p-2^j} + x_{p+2^j} - 2x_p\|/4^{i+1} \).

Prove that \( \sum_{(p, j) \in A} \lambda(\bar{x}, p, j)^2 \leq \sum_{j=0}^{n-1} \|x_{j+1} - x_j\|^2 \).

(c) Now suppose \( f \) is a non-expansive map from \( T_n \) into \( \ell_2 \).

An equi-triple is a triple \((u, v, w)\) of vertices where \( u \prec v \prec w \) and \( d_T(v, w) = d_T(u, v) \). For an equi-triple \((u, v, w)\) define \( \delta(u, v, w) = \|f(u) - 2f(v) + f(w)\|/d_T(u, v) \).

A fork is a quadruple \((u, v, w, w')\) where \((u, v, w)\) and \((u, v, w')\) are both equi-triples and \( v \) is the least common ancestor of \( w, w' \). Show that if \((u, v, w, w')\) is any fork then the distortion of \( f \) is at least \( 1/(\delta(u, v, w) + \delta(u, v, w')) \).

(d) For a binary string \( \beta \) of length \( n \) and \( 0 \leq j \leq n \) let \( x_j^\beta = f(\beta(j)) \) and let \( \bar{x}^\beta \) be the sequence \( x_0^\beta, \ldots, x_n^\beta \). For each \((p, j) \in A\), express \( \sum_{\beta \in \{0, 1\}^n} \lambda(\bar{x}^\beta, p, j)^2 \) as a linear combination of terms \( \|\delta(u, v, w)\|^2 + \|\delta'(u, v, w')\|^2 \), where \((u, v, w, w')\) ranges over some set of forks.

(e) Finish the proof of the theorem.