# Workshop 3 – Limits and Continuity

1. Let $f(x) = (1+x)^3 - 1$.

   (a) Find the limit $L = \lim_{x \to 0} f(x)$.

   (b) Prove that your answer to (a) is correct using the limit laws.

   (c) Give another proof of your answer to (a), this time using the $\varepsilon$-$\delta$ definition of a limit.

2. For each of the three cases below, sketch a graph of a function that satisfies the stated conditions. In each case, the *domain* of the function should be *all real numbers.*

   (a) $\lim_{x \to 2} f(x) = 3$ and $f(2) = 4$.

   (b) $\lim_{x \to 0} f(x)$ does not exist, and $|f(x)| < 2$ for all $x$.

   (c) $\lim_{x \to 1} f(x)$ exists and its value is $f(1) + 2$.

   (d) $\lim_{x \to -1^-} f(x)$ and $\lim_{x \to -1^+} f(x)$ do not exist, $|f(x)| < 3$ for all $x$, and $f(-1) = -2$.

3. A rational function is a quotient of two polynomials. Simple examples of rational functions are $x+1$ and $\dfrac{(3x+7)(6x-1)}{(x^2+6)(x-11)}$ and $\dfrac{2.03x^{78} + 3.7x^{45} - .09}{4.5x^{99} - \sqrt{13}x}$ and $\dfrac{x-2}{x-2}$.

   (a) (Easy) Find a rational function $R$ whose domain is all real numbers *except* 1 and 2, and so that for all $x$ in its domain, $R(x) = x^2$. Sketch a graph of $R$.

   (b) (Harder) Find a rational function $T$ with *all* of these properties. Sketch a graph of $T$.

      i. The natural domain of $T$ is all real numbers *except* for 1 and 3 and 4.

      ii. $\lim_{x \to 1^-} T(x) = +\infty$ and $\lim_{x \to 1^+} T(x) = +\infty$;
      $\lim_{x \to 3^-} T(x) = -\infty$ and $\lim_{x \to 3^+} T(x) = +\infty$;
      $\lim_{x \to 4^-} T(x) = +\infty$ and $\lim_{x \to 4^+} T(x) = -\infty$;
      $\lim_{x \to +\infty} T(x) = +\infty$ and $\lim_{x \to -\infty} T(x) = +\infty$.

      iii. $T$ has roots only at 0 and 2 and 5.

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4. Let

$$f(x) = \begin{cases} 3x-2, & \text{if } x < 0 \\ ax+b, & \text{if } 0 \le x \le 1 \\ 3x+4, & \text{if } x > 1 \end{cases}$$

Find $a$ and $b$ so that $f(x)$ is continuous for all values of $x$.

5. (a) Suppose $F(x) = x^3$ and $G(x) = 4\cos(7x+5) + 8\sin(x^2-9) + 6$. What are $\lim_{x\to+\infty} F(x)$ and $\lim_{x\to-\infty} F(x)$? Find numbers $A$ and $B$ so that all values of $G$ are between $A$ and $B$ (that is, $A < G(x) < B$ for all $x$). The values of $A$ and $B$ don't have to be precise!

   (b) Make a sketch on the same graph of $y = F(x)$ and $y = G(x)$ and $y = A$ and $y = B$.

   (c) Find an approximate value of one root of $F(x) = G(x)$.

   (d) Prove the following result: **if** $F$ and $G$ are continuous functions defined on $\mathbb{R}$ (all real numbers) and **if** $\lim_{x\to+\infty} F(x) = +\infty$ and $\lim_{x\to-\infty} F(x) = -\infty$ and **if** $G$ is bounded (so there are numbers $A$ and $B$ so that $A < G(x) < B$ for all $x$) then the equation $F(x) = G(x)$ **must** have at least one root. (Hint: Look up the Intermediate Value Theorem.)

Please write solutions to ONE of these problems (as announced in the Workshop) and hand it in at the Workshop next week. Your written solution should follow the guidelines posted on the web at `http://www.math.rutgers.edu/~sasar/151/guidelines.php`. You may discuss the problems with other students and with your instructors, but the written work you hand in must be your own.
DO NOT TRY TO WORK OUT THE SOLUTIONS ON THIS PIECE OF PAPER. Bring a notebook to workshop to record your calculations during the workshop. These notes can then be the starting-point (but not the finishing point) for the writeup that you hand in.