HW1 SOLUTION HINTS

There is always more than one way to solve a problem so please feel free to ignore these hints. Also note that below we give sketches only, not complete solutions. The intension is that you could use them to easily write complete solutions by filling in the gaps.

1.1. *If x is irrational and r is nonzero and rational then $rx, r+x$ are rational.*

Proof. If $p := r + x \in \mathbb{Q}$ then since $\mathbb{Q}$ is a field, $x = p - r \in \mathbb{Q}$, contradiction. The case of rx is similar. $\square$

1.2. *$\sqrt{12}$ is irrational.*

Proof. If $\sqrt{12}$ is rational, then $\sqrt{3}$ is rational, since $p^2 = 12$ implies $(p/2)^2 = 3$. Hence it suffices to show that $\sqrt{3}$ is irrational, and this can be done using the same method as the one we used in class to show $\sqrt{2}$ is irrational. $\square$

1.4. *Suppose that S is linearly ordered and $\emptyset \neq E \subset S$. If α is a lower bound for E and β is an upper bound for E, then $\alpha \leq \beta$.*

Proof. Since $E \neq \emptyset$, we may let $x \in E$ be arbitrary. Since α is a lower bound of E, $\alpha \leq x$. Since β is an upper bound of E, $x \leq \beta$. Hence $\alpha \leq \beta$. $\square$

1.5. *Suppose $A \subset \mathbb{R}$ is bounded below. Then $\inf A = -\sup(-A)$.*

Proof. Let S be the set of lower bounds for A. Since α is a lower bound for A iff $-\alpha$ is an upper bound for $-A$, we have that $-S$ is the set of upper bounds for $-A$. Clearly, if α is the greatest element of S, then $-\alpha$ is the least element of $-S$, which completes the proof. $\square$

1.6. *Let $b > 1$.*

(a) *If $m/n = p/q$, show that $(b^m)^{1/n} = (b^p)^{1/q}$.*
(b) *If $r, s \in \mathbb{Q}$, show that $b^{r+s} = b^r b^s$.*
(c) *If $r \in \mathbb{Q}$, show that $b^r = \sup\left\{b^t : t \in \mathbb{Q}, t \leq r\right\}$.*
(d) *Define $b^x = \sup\left\{b^t : t \in \mathbb{Q}, t \leq x\right\}$ and show that $b^{x+y} = b^x b^y$.*

Proof. (a). Notice $mq = pn$, so that $b^{mq} = b^{pn}$. Taking the nq^{th} root of both sides and rearranging the parentheses (which must be justified), we get

$$(((b^m)^q)^{1/q})^{1/n} = (((b^p)^n)^{1/n})^{1/q}$$

Appealing to the definition of n^{th} root and q^{th} root yields $(b^m)^{1/n} = (b^p)^{1/q}$.

(b). If now $r = m/n$ and $s = p/q$ then $b^{r+s} = b^{(mq+np)/nq}$ and $b^r b^s = b^{m/n} b^{p/q}$. Take both expressions to the nq and appeal to Corollary 1.21.

(c). It suffices to show that for $r, s \in \mathbb{Q}$, $r < s$ implies $b^r < b^s$, since then b^r is clearly the maximum of $\left\{b^t : t \in \mathbb{Q}, t \leq r\right\}$. But this follows from (b) since if $r < s$ then s can be written as $r + t$ with $t > 0$, and then $b^s = b^r b^t > b^r$.

(d). Let $\delta > 0$, and choose $r, s \in \mathbb{Q}$ such that $r < x$, $s < y$, $b^x < b^r + \delta$, and $b^y < b^s + \delta$. Then using (c):

$$\begin{aligned} b^x b^y &\leq (b^r + \delta)(b^s + \delta) \\ &= b^r b^s + \delta(-junk-) \\ &\leq b^{r+s} + \delta(-junk-) \\ &\leq b^{x+y} + \delta(-junk-) \end{aligned}$$

Since δ can be arbitrarily small, we have $b^x b^y \leq b^{x+y}$. For the reverse inequality, let $\delta > 0$ and choose $t < x + y$ such that $b^{x+y} \leq b^t + \delta$. Then write $t = r + s$ where $r, s \in \mathbb{Q}$, $r < x$, and $s < y$ (this requires justification using rational density!). Again using (c):

$$\begin{aligned} b^{x+y} &\leq b^t + \delta \\ &= b^{r+s} + \delta \\ &= b^r b^s + \delta \\ &\leq b^x b^y + \delta \end{aligned}$$

Again since δ was arbitrary, we must have $b^{x+y} \leq b^x b^y$. ☐

1.8. *There is no ordering on* $\mathbb{C}$ *which makes it into an ordered field.*

Proof. Suppose that $(\mathbb{C}, <)$ is an ordered field. By 1.18(d) both $1 = 1^2$ and $-1 = i^2$ are positive (with respect to $<$). But 1.18(a) implies that 1 and -1 have opposite signs, a contradiction. ☐