HW2 SOLUTION HINTS

1.9. *If $z = a + bi, w = c + di$, then define $z < w$ if $a < c$, or else $a = c$ and $b < d$. Prove that this is a linear ordering of $\mathbb{C}$ and decide if it has the LUPB.*

Proof. Thinking of z, w as points on the plane, the definition of $<$ means that $z < w$ iff z lies to the west of w (but north and south as much as it wants), or else *due south* of w.

To show this ordering satisfies trichotomy (that is, for any z, w either $z = w$, $z < w$, or $w < z$), just think of configurations of two points in the plane. If one of them lies to the east of the other (ignoring the north-south distance), then that point is $<$ the other. If neither lies east or west of the other, then they must lie in the same vertical line. But in this case, one lies due south of the other, and this one is $<$ than the other.

Transitivity (that is, for any z, w, v, if $z < w$ and $w < v$ then $z < v$) is a bit harder because you must consider the possible configurations of *three* points in the plane, and so there are more cases. I'll omit this.

As for the LUPB, the answer is no. Think of $S =$ the set of points on the y-axis. This set is bounded above. In fact, the set of upper bounds for S is precisely the set of points to the right of the y-axis. But S has no *least* upper bound, since every point to the right of the y-axis has points due south of it, which are also to the right of the y-axis. □

1.11. *If $z \in \mathbb{C}$, show there exists $r \geq 0$ and $w \in \mathbb{C}$ with $|w| = 1$ such that $z = rw$. Decide whether r, w are uniquely determined by z.*

Proof. Thinking of r as the length of z, and w as the direction, we let $r = |z|$ and $w = z/r$. If $r = 0$, then this doesn't make sense but in that case we take w to be anything of unit length. Clearly we have that $z = rw$.

The choice of r is unqiue, since taking absolute values on both sides of $z = rw$ implies $r = |z|$. We have seen that the choice of w is not unique if $r = 0$ (*i.e.*, z=0). But if $r \neq 0$, then $z = rw$ implies $w = z/r$, so this is the only choice. □

1.12. *If $z_1, \ldots, z_n \in \mathbb{C}$, prove that $|z_1 + \cdots + z_n| \leq |z_1| + \ldots + |z_n|$.*

Proof. The case $n = 1$ is trivial, and the case $n = 2$ is just the ordinary triangle inequality. To prove it in general, suppose that $z_1, \ldots, z_{n+1}$ are given and suppose inductively that $|z_1 + \cdots + z_n| \leq |z_1| + \ldots + |z_n|$. Then:

$$\begin{aligned} |z_1 + \cdots + z_{n+1}| &\leq |z_1 + \cdots + z_n| + |z_{n+1}| && \text{(by the triangle inequality)} \\ &\leq |z_1| + \ldots + |z_n| + |z_{n+1}| && \text{(by inductive hypothesis)} \end{aligned}$$

and this concludes the proof. □

1.13. *If $x, y \in \mathbb{C}$, prove that $||x| - |y|| \leq |x - y|$.*

Proof. This is really just a rephrasing of the triangle inequality. Consider the triangle with vertices $0, x, y$. Then the sides have length $|x|, |y|, |x-y|$. The journey from 0 to x is faster than going through y as a waypoint, so

$$|x| \leq |y| + |x - y|$$

Similarly, the journey from 0 to y is faster than using x as a waypoint, so

$$|y| \leq |x| + |x - y|$$

Rewriting these, we get that $|x - y|$ is greater than or equal to both $|x| - |y|$ and $|y| - |x|$, hence it is greater than or equal to $||x| - |y||$. □

1.17. *If $x, y \in \mathbb{R}^k$, prove that $|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2$. What does this say geometrically about parallelograms?*

Proof.

$$\begin{aligned} |x+y|^2 + |x-y|^2 &= (x+y)\cdot(x+y) + (x-y)\cdot(x-y) \\ &= (x\cdot x) + 2(x\cdot y) + (y\cdot y) + (x\cdot x) - 2(x\cdot y) + (y\cdot y) \\ &= 2|x|^2 + 2|y|^2 \end{aligned}$$

Just draw a picture, and see that this is saying the sum of the squares of the sides of a parallelogram equals the sum of the squares of the diagonals. Our proof is algebraic and doesn't show why this should be true. Can you prove it geometrically? □

1.18. *If $k \geq 2$ and $x \in \mathbb{R}^k$, prove that there exists $y \in \mathbb{R}^k$ such that $x \cdot y = 0$. What about $k = 1$?*

Proof. If $k = 2$ then let $x = (x_1, x_2)$ be given. If we let $y = (-x_2, x_1)$ then

$$x \cdot y = x_1(-x_2) + x_2 x_1 = 0$$

as desired. To generalize this to $k > 2$, suppose that $x = (x_1, x_2, \ldots, x_k)$. Then just let $y = (-x_2, x_1, 0, \ldots, 0)$. □