HW3 SOLUTION HINTS

2.2. *Prove that the set of algebraic numbers is countable.*

Proof. The main claim is that the set of polynomials with integer coefficients is countable. Admitting this, since every polynomial has finitely many roots (the fundamental theorem of algebra), we will have that the algebraic numbers is a countable union of finite sets, and hence countable.

To see that there are countably many polynomials, observe that there each polynomial of degree n is determined by its $n+1$ coefficients from $\mathbb{Z}$. Hence if P_n denotes the set of polynomials of degree n with integer coefficients, then P_n has the same size as $\mathbb{Z}^{n+1}$. We showed in class that $\mathbb{Z}$ and $\mathbb{Z}^2$ are countable, and it is easy to use induction to conclude that $\mathbb{Z}^n$ is countable for every n. Hence, P_n is countable for every n. Finally, the set of all polynomials with integer coefficients is $\bigcup P_n$, which is a union of countable sets and hence countable. □

2.3. *Prove that there exist real numbers which are not algebraic.*

Proof. By the last exercise, only countably many of the real numbers are algebraic. Since there are uncountably many real numbers, some of them must be non-algebraic. □

2.4. *Is the set of irrational numbers countable?*

Proof. If the set $\mathbb{I}$ of irrational numbers were countable, then since $\mathbb{Q}$ is countable, we would have that $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$ is countable. This is a contradiction, so we must have that $\mathbb{I}$ is uncountable. □

2.5. *Construct a bounded set of reals with exactly three limit points.*

Proof. The set $S = \{1/n : n \in \mathbb{N}\} \cup \{1 + 1/n : n \in \mathbb{N}\} \cup \{2 + 1/n : n \in \mathbb{N}\}$ will do. □

2.6. *Let E' denote the set of all limit points of E. Prove that E' is closed. Prove that E and $\bar{E}$ have the same limit points. Do E and E' always have the same limit points?*

Proof. We'll give the answer this Thursday. □

2.11. *Check whether the following are metrics on* $\mathbb{R}$*:*

(a) $d_1(x, y) = (x - y)^2$
(b) $d_2(x, y) = \sqrt{|x - y|}$
(c) $d_3(x, y) = |x^2 - y^2|$
(d) $d_4(x, y) = |x - 2y|$
(e) $d_5(x, y) = |x - y| / (1 + |x - y|)$

Answers. (a) No, since $d_1(0, 2) = 4$ but $d_1(0, 1) + d_1(1, 2) = 1 + 1 = 2$.

(b) Yes. We want to show that

$$\sqrt{|x-z|} \leq \sqrt{|x-y|} + \sqrt{|y-z|}$$

Taking the square of both sides, it is enough to show that

$$|x-z| \leq |x-y| + 2\sqrt{|x-y| \cdot |y-z|} + |y-z|$$

Throwing away the $2\sqrt{\cdots}$ term, the right-hand side only becomes smaller, and then it of course boils down to the usual triangle inequality.

(c) No, since $d_3(-5, 5) = 0$.

(d) No, since $d_4(x, y) \neq d_5(y, x)$.

(e) Yes, but the triangle inequality is a bit of work. I omit it.

□