## HW4 SOLUTION HINTS

**2.6.** *Let $E'$ denote the set of all limit points of $E$. Prove that $E'$ is closed. Prove that $E$ and $\bar{E}$ have the same limit points. Do $E$ and $E'$ always have the same limit points?*

*Proof.* One way to show that a set is closed is to show it contains all its limit points. For this we use a two-step process, see Figure 1. So let $x$ be a limit point of $E'$, that is, a point of $E''$. We must show that every neighborhood of $x$ contains points of $E$ (other than $x$). So let $N_r(x)$ be an arbitrary neighborhood of $x$. Since $x \in E''$, we know that there exists $y \in N_r(x) \cap E'$ such that $y \neq x$. Since $y \in E'$, every neighborhood of $y$ contains points of $E$ other than $y$. Letting $\delta = \min\{d(x,y), r - d(x,y)\}$, there exists $z \in N_\delta(y) \cap E$ such that $z \neq y$. By choice of $\delta$, we have that $z \neq x$ either. Then since

$$d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + r - d(x,y) = r$$

we have that $z \in N_r(x)$ and $z \in E$. This concludes the proof.

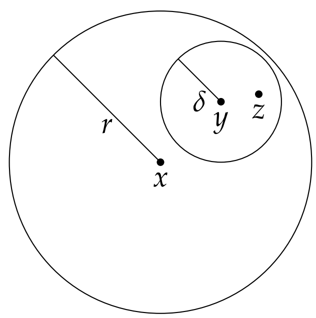


FIGURE 1. A two-step process: If $x \in E''$ then we can find a nearby $y \in E'$, and if $y \in E'$ then we can find a nearby $z \in E$. Using the triangle inequality, $x$ is near to $z$.

Since $E \subset \bar{E}$, it is trivial to see that every limit point of $E$ is a limit point of $\bar{E}$. Conversely, if $x$ is a limit point of $\bar{E}$, we must show that $x$ is a limit point of $E$. That is, we must show that every neighborhood of $x$ contains points of $E$ other than $x$ itself. So let $N_r(x)$ be an arbitrary neighborhood of $x$. Then $N_r(x)$ contains a point $y \neq x$ of $\bar{E}$. Since $\bar{E} = E \cup E'$, we must have either $y \in E$ or $y \in E'$. If $y \in E$, then we are done. If $y \in E'$, then we do exactly the same thing as in the previous paragraph to find a $z \in E$ such that $z \neq x$ and $z \in N_r(x)$.

Finally, $E$ and $E'$ do not always have the same limit points. For instance, if $E = \{1/n : n \in \mathbb{N}\}$ then $E' = \{0\}$, but $E'' = \{0\}' = \emptyset$. ☐

**2.8.** *Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of $E$? Is every point of every closed set $E \subset \mathbb{R}^2$ a limit point of $E$?*

*Answers.* The answer to the first question is "yes." If $E \subset \mathbb{R}^2$ is open and $x \in E$, then there exists a disc $D_r(x)$ centered at $x$ such that $D_r(x) \subset E$. Now, if $D'$ is any other disc centered at $x$, then either $D' \cap D_r(x)$ contains infinitely many points and is moreover contained in $E$. Hence $D'$ contains infinitely many points of $E$.

By the way, this argument works in $\mathbb{R}^2$ because neighborhoods are infinite. It will not work in a discrete space such as $\mathbb{Z}^2$, where neighborhoods can be finite and in fact every point is isolated.

The answer to the second question is "no," since for instance a singleton set $\{x\}$ is closed but has no limit points. □

**2.9.** *Let* $\text{int}(E)$ *denote the set of interior points of $E$.*

(a) *Prove that* $\text{int}(E)$ *is open.*
(e) *Do $E$ and $\bar{E}$ always have the same interior?*
(f) *Do $E$ and* $\text{int}(E)$ *always have the same closure?*

*Proof.* (a) If $x \in \text{int}(E)$, then there exists a neighborhood $N_r(x)$ which is contained in $E$. To show $x$ is an interior point of $\text{int}(E)$, it suffices to show that $N_{r/2}(x)$ is contained in $\text{int}(E)$. But this is easy, since if $y \in N_{r/2}(x)$ then $N_{r/2}(y) \subset N_r(x) \subset E$ (see Figure 2), so that $y \in \text{int}(E)$.

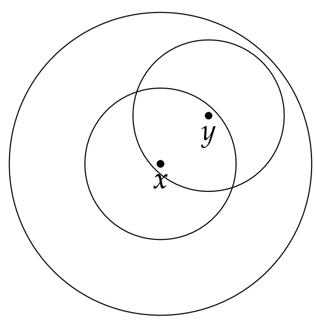


FIGURE 2. The larger circle has radius $r$, the two smaller circles have radius $r/2$. The picture shows that $N_{r/2}(y) \subset N_r(x)$.

(e) The answer is "no." One example is the punctured disc $D^* = \{x \in \mathbb{R}^2 : 0 < |x| < 1\}$. Since it is open, $\text{int}(D^*) = D^*$. But $\overline{D^*}$ is the closed disc $\{x \in \mathbb{R}^2 : |x| \leq 1\}$. Hence $\text{int}(\overline{D^*}) =$ the open disc $\{x \in \mathbb{R}^2 : |x| < 1\}$.

(f) The answer is "no." Again use a singleton $\{x\}$ in the plane. The closure is $\{x\}$, but the interior is $\emptyset$ so the closure of the interior is $\emptyset$. □

**2.10.** *Let $X$ be an infinite set, with metric $d(x,y) = 1$ iff $x \neq y$ and $d(x,x) = 0$.*

(a) *Prove that this is a metric.*

(b) *Which subsets of X are open? Closed?*

*Proof.* (a) The only nontrivial property to check is the triangle inequality. The only interesting case of the triangle inequality is the case of three distinct points $x, y, z$. Then $d(x,z) = 1$ which is smaller than $d(x,y) + d(y,z) = 2$.

(b) Every subset of $X$ is open. To see this, let $A \subset X$ and let $x \in A$. Then $N_{1/2}(x) = \{x\} \subset A$ so that $x \in \text{int}(A)$.

It follows that every subset of $X$ is closed, since the closed sets are just the complements of the open sets, and every set is open. □

**2.22.** *Show that $\mathbb{R}^n$ is separable, that is, has a countable dense set.*

*Proof.* We proved this for $\mathbb{R}^2$ in class, here is the generalization. Suppose that $\boldsymbol{x} \in \mathbb{R}^n$ and let $r > 0$. We will show that $\mathbb{Q}^n$ is dense, that is, there exists $\boldsymbol{q} \in \mathbb{Q}^n$ such that $\boldsymbol{q} \in N_r(\boldsymbol{x})$. By rational density, for each $i \leq n$, we may choose $q_i \in \mathbb{Q}$ such that $|x_i - q_i| < \frac{r}{n}$. For each $i \leq n$, consider the vector

$$\boldsymbol{y}^i = (q_1, \ldots q_i, x_{i+1}, \ldots x_n)$$

Then repeatedly applying the triangle inequality, we have

$$\begin{aligned} d(\boldsymbol{x}, \boldsymbol{q}) &\leq d(\boldsymbol{x}, \boldsymbol{y}^1) + d(\boldsymbol{y}^1, \boldsymbol{y}^2) + \cdots + d(\boldsymbol{y}^{n-1}, \boldsymbol{q}) \\ &< \frac{r}{n} + \frac{r}{n} + \cdots + \frac{r}{n} \\ &= r \end{aligned}$$

Finally, note that $\mathbb{Q}^n$ is countable since any finite product of countable sets is countable. This completes the proof. □

**2.23.** *Prove that any separable space has a countable base, that is, there exists a countable collection $V_\alpha$ of open sets such that for any open $G$ and $x \in G$ we have $x \in V_\alpha \subset G$ for some $\alpha$.*

*Proof.* Let $D$ be a countable dense set. We consider the family of sets $N_q(d)$ where $0 < q \in \mathbb{Q}$ and $d \in D$. This family is countable since $D \times \mathbb{Q}$ is a product of countable sets and hence countable.

To see that this is a base, let $G$ be open and $x \in G$. Since $x$ is an interior point, there exists $r$ such that $N_r(x) \subset G$. Since $D$ is dense there exists $d \in D$ such that $d \in N_{r/2}(x)$. Let $q$ be any rational number such that $d(x,d) < q < r/2$. Since $q < r/2$, we have that $N_q(d) \subset N_r(x)$ (this is just the same picture as Figure 2). Since $d(x,d) < q$, we have $x \in N_q(d)$. We have shown that $x \in N_q(d) \subset N_r(x) \subset G$, which completes the proof. □