HW5 SOLUTION HINTS

2.13. *Construct a compact set of real numbers whose limit points form a countable (infinite) set.*

Proof. There are many ways to do this, here is just one example. We know that $\{0\} \cup \{1/n\}$ is a compact set with just one limit point. My idea is to try to squeeze infinitely many sets like this into a bounded interval. So define

$$S_m = \left\{ \frac{1}{2^m} \right\} \cup \left\{ \frac{1}{2^m} + \frac{1}{2^m}\frac{1}{n} \right\}$$

Each S_m has one limit point, and all the S_m lie in disjoint intervals, since S_m is actually contained in the interval $[1/2^m, 2/2^m)$. Thus the limit points don't "interfere" with each other in any way.

Now, let $S = \{0\} \cup \bigcup S_m$. It is easy to check that S' is precisely $\{0\} \cup \{1/2^m\}$. Then S' is countably infinite, and moreover $S' \subset S$ so S is closed. Of course, S is also bounded, and hence it is compact. □

2.24. *Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable.*

Proof. Following the hint, we first claim that for any $\delta > 0$, the space X can be covered by finitely many balls of radius δ. Beginning with any x_1, choose $x_2 \notin N_\delta(x_1)$, then choose $x_3 \notin N_\delta(x_1) \cup N_\delta(x_2)$. In general, choose $x_{n+1} \notin N_\delta(x_1) \cup \cdots \cup N_\delta(x_n)$ if it is possible to do so. Note that this process must in fact halt at some stage n, since otherwise the sequence $\{x_i\}$ would be an infinite set which is uniformly discrete. Such a set cannot have any limit points, but this contradicts the hypothesis that every infinite subset of X has a limit point. Once the process stops, we conclude that the neighborhoods $N_\delta(x_1), \ldots, N_\delta(x_n)$ cover X, completing the claim.

Now, for each n, cover X by finitely many neighborhoods of radius $\delta = 1/n$ and let D_n be the *centers* of these neighborhoods. Then $D = \bigcup D_n$ is a countable union of finite sets, and hence it is countable. To see that D is dense, let $x \in X$ and ϵ be arbitrary; we wish to show that there exists $y \in D$ such that $d(x, y) < \epsilon$. To do this, choose n large enough so that $1/n < \epsilon$. Since the neighborhoods $N_{1/n}(y)$ for $y \in D_n$ cover X, there must exist a particular $y \in D_n$ such that $x \in N_{1/n}(y)$. Thus $d(x, y) < 1/n < \epsilon$, as desired. Thus D is a dense subset of X, and hence X is separable. □