HW8 SOLUTION HINTS

3.6. *Determine whether the following series converge or diverge.*

- $\sum(\sqrt{n+1}-\sqrt{n})$
- $\sum(\sqrt{n+1}-\sqrt{n})/n$
- $\sum(\sqrt[n]{n}-1)^n$

Answer. For the first:

$$\begin{aligned}\sum(\sqrt{n+1}-\sqrt{n}) &= \sum\frac{\sqrt{n+1}-\sqrt{n}}{1}\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \\ &= \sum\frac{1}{\sqrt{n+1}+\sqrt{n}} \\ &\geq \frac{1}{2}\sum\frac{1}{\sqrt{n+1}}\end{aligned}$$

The last is a divergent p-series, so the original series must diverge.

For the second:

$$\begin{aligned}\sum\frac{\sqrt{n+1}-\sqrt{n}}{n} &= \sum\frac{\sqrt{n+1}-\sqrt{n}}{n}\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \\ &= \sum\frac{1}{n(\sqrt{n+1}+\sqrt{n})} \\ &\leq \frac{1}{2}\sum\frac{1}{n\sqrt{n}}\end{aligned}$$

The last is a convergent p-series, so the original series must converge.

Finally, for the last problem we apply the root test. The n^{th} root of the n^{th} term is $\sqrt[n]{n}-1$, and this converges to 0 by 3.20 in the book. Hence, the root test implies that this series converges. □

3.7. *If $\sum a_n$ is convergent, show that $\sum\sqrt{a_n}/n$ is convergent.*

Proof. There is no easy way to do this. One way is to notice that $\sqrt{a_n}/n$ is either less than $1/n^2$ or else less than a_n. Another way is to notice that since

$$0\leq(\sqrt{a_n}-1/n)^2 = a_n - 2\sqrt{a_n}/n + 1/n^2$$

we then have

$$2\sqrt{a_n}/n \leq a_n + 1/n^2$$

And hence

$$\sum \sqrt{a_n}/n \leq \frac{1}{2}\sum(a_n + \frac{1}{n^2})$$
$$= \frac{1}{2}\sum a_n + \frac{1}{2}\sum \frac{1}{n^2}$$

which is a sum of two convergent series. □

3.8. *Suppose that $\sum a_n$ is convergent and b_n is bounded and monotonic. Prove that $\sum a_n b_n$ is convergent.*

Proof. Once again, this problem is too hard. The best way to do it is to use a theorem in the book which says that if the partial sums of $\sum a_n$ are bounded and b_n decreases to 0 then $\sum a_n b_n$ is convergent. Since $\sum a_n$ is convergent, we have the first hypothesis.

To get the second hypothesis, let us suppose without loss of generality that b^n is increasing (if it is decreasing, you can just negate everything). Then since b_n is increasing and bounded, it converges to its supremum, let us call it b. Now,

$$\sum a_n b_n = b \sum a_n - \sum a_n(b - b_n)$$

The first term $b\sum a_n$ converges, and now since $b - b_n$ is decreasing to 0, the second term $\sum a_n(b - b_n)$ converges by the theorem in the book! □

3.19. *Prove that the Cantor set is precisely the set of sums $\sum a_n/3^n$ where all coefficients a_n are either 0 or 2.*

Proof. I don't want to go into too much detail here. The point is to think of elements of $[0,1]$ by their ternary expansion. Then, insisting that $a_1 \neq 1$ amounts to omitting the middle third. Next, insisting that $a_2 \neq 1$ amounts to omitting the middle thirds of each third. And so on. . . . □