MATH 351/641 MIDTERM ONE SOLUTIONS

Instructions:

- All students: choose **four** of the **first five** problems.
- Graduate students: additionally, choose **one** of problems **six and seven**.

Problem 1. Let A be a set of real numbers which is bounded above and let $\alpha = \sup(A)$. Show that if $\alpha \notin A$ then α is a limit point of A.

Proof. If $\alpha \notin A'$ then there exists a neighborhood $N_\epsilon(\alpha)$ which contains no point of A other than possibly α. But since $\alpha \notin A$, in fact $N_\epsilon(\alpha)$ contains no point of A whatsoever. It follows that $\alpha - \epsilon/2$ is an upper bound for A, contradicting that α was the *least* upper bound of A. □

Problem 2. Show that if S is a closed set of real numbers then S has the LUBP, *i.e.*, every bounded subset of S has a least upper bound in S. (Hint: use Problem 1.)

Proof. If $A \subset S$ is a bounded subset and $\alpha = \sup(A)$, we must show that $\alpha \in S$. We use two cases: First, if $\alpha \in A$, then since $A \subset S$, of course $\alpha \in S$ as well. Second, if $\alpha \notin A$, then by the previous problem $\alpha \in A'$. But clearly since $A \subset S$ we have $A' \subset S'$, so it follows that $\alpha \in S'$. Finally, since S is closed, it contains its limit points and so $\alpha \in S$. □

Problem 3. Let S be the set of finite strings of letters A–Z. Prove that S is countable.

Proof. For each $n \in \mathbb{N}$, let S_n be the set of such strings of length exactly n. Then it is easy to see that there are exactly 26^n elements of S_n, in particular S_n is finite. Now $S = \bigcup S_n$ is a countable union of finite sets, and hence it is countable. □

Problem 4. Prove or give a counterexample to the following statement: If X is a metric space and $A, B \subset X$, then $\partial(A \cup B) = \partial A \cup \partial B$.

Proof. This is false. For example work in $X = \mathbb{R}$ and let $A = [0,1]$ and $B = [0,2]$. Then $\partial A = \{0,1\}$ and $\partial B = \{0,2\}$ so that $\partial A \cup \partial B = \{0,1,2\}$. But $A \cup B = [0,2]$ so that $\partial(A \cup B) = \{0,2\}$ which is not the same. □

Problem 5. Consider the following metric on $\mathbb{R}^2$:

$$d((x_1,x_2),(y_1,y_2)) = \max\{|x_1-y_1|,|x_2-y_2|\}$$

(a) Prove that d satisfies the triangle inequality. (Hint: you can use the ordinary triangle inequality.)
(b) What shape do the d-neighborhoods have? (No proof required.)

Proof. Let $(x_1,x_2),(y_1,y_2),(z_1,z_2)$ be any three points in the plane. We wish to show that

$$\max\{|x_1-z_1|,|x_2-z_2|\} \\ \leq \max\{|x_1-y_1|,|x_2-y_2|\} + \max\{|y_1-z_1|,|y_2-z_2|\}$$

For this, it suffices to check that both $|x_1-z_1|$ and $|x_2-z_2|$ are less than or equal to the right-hand side.

$$\begin{aligned} |x_1-z_1| &\leq |x_1-y_1|+|y_1-z_1| \\ &\leq \max\{|x_1-y_1|,|x_2-y_2|\} + \max\{|y_1-z_1|,|y_2-z_2|\} \end{aligned}$$

and

$$\begin{aligned} |x_2-z_2| &\leq |x_2-y_2|+|y_2-z_2| \\ &\leq \max\{|x_1-y_1|,|x_2-y_2|\} + \max\{|y_1-z_1|,|y_2-z_2|\} \end{aligned}$$

For part (b), the d-neighborhoods all have the shape of a square. □

Problem 6. Let (X,d) be a metric space, and let $Y \subset X$. Consider Y as a metric space in its own right, with the same metric d. Show that if $A \subset Y$ is open then there exists an open $A' \subset X$ such that $A = A' \cap Y$.

We will use the following notation: If $y \in Y$, write write

$$N_r^X(y) = \{z \in X : d(y,z) < r\}$$

and

$$N_r^Y(y) = \{z \in Y : d(y,z) < r\}$$

Proof. Since A is open in Y, for each $a \in A$ there exists a radius r_a such that the neighborhood $N_{r_a}^Y(a) \subset A$. Clearly, A is exactly the union $\bigcup_{a \in A} N_{r_a}^Y(a)$. We simply let $A' := \bigcup_{a \in A} N_{r_a}^X(a)$.

- Since A' is a union of neighborhoods of X, it is clearly open in X.
- Since each $N_{r_a}^Y(a) \subset N_{r_a}^X(a)$, we have that $A \subset A' \cap Y$.
- Finally, if $z \in A' \cap Y$ then there is some $a \in A$ such that $z \in N_{r_a}^X(a)$. Since $z \in Y$, it follows that $z \in N_{r_a}^Y(a)$, and hence $z \in A$. Thus $A' \cap Y \subset A$.

We have shown that A' is open and $A = A' \cap Y$. □

Problem 7. Suppose that (X_1, d_1) and (X_2, d_2) are separable metric spaces. Give $X_1 \times X_2$ the usual metric

$$d((x_1, x_2), (y_1, y_2)) = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2}\,.$$

Prove that the space $(X_1 \times X_2, d)$ is separable. (Hint: generalize the proof that $\mathbb{R}^2$ is separable.)

Proof. Let D_1 be a countable dense subset of X_1 and let D_2 be a countable dense subset of X_2. Then $D_1 \times D_2$ is countable, and we will show that $D_1 \times D_2$ is a dense subset of $X_1 \times X_2$. That is, given $(x_1, x_2) \in X_1 \times X_2$ and $\epsilon > 0$, we must find an element $(a_1, a_2) \in D_1 \times D_2$ within ϵ of (x_1, x_2).

First, use the density of D_1 to find $a_1 \in D_1$ such that $D_1(x_1, a_1) < \epsilon/2$. Second, use the density of D_2 to find $a_2 \in D_2$ such that $d_2(x_2, a_2) < \epsilon/2$. Then $(a_1, a_2) \in D_1 \times D_2$ and:

$$\begin{aligned} d((x_1, x_2), (a_1, a_2)) &\le d((x_1, x_2), (a_1, x_2)) + d((a_1, x_2), (a_1, a_2)) \\ &= \sqrt{d_1(x_1, a_1)^2 + \cancel{d_2(x_2, x_2)^2}} + \sqrt{\cancel{d_1(a_1, a_2)^2} + d_2(x_2, a_2)^2} \\ &= d_1(x_1, a_1) + d_2(x_2, a_2) \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

□

Here is the idea in a picture: