MATH 351/641 MIDTERM TWO SOLUTIONS

Instructions:

- All students: choose **four** of the **first five** problems.
- Graduate students: additionally, choose **one** of problems **six and seven**.

Problem 1. Suppose that X is a metric space and $A, B \subset X$ are compact. Prove that $A \cup B$ is compact. Please use the "metric space" definition of compcatness—partial credit will be given if you use the equivalent definition for subsets of $\mathbb{R}^n$.

Proof. Let S be an infinite subset of $A \cup B$, we must show that S has a limit point in $A \cup B$. To see this, consider the sets $S_1 = S \cap A$ and $S_B = S \cap B$. Since S is infinite and $S = S_1 \cup S_2$, we must have that either S_1 or S_2 is infinite. If S_1 is infinite, then since $S_1 \subset A$ and A is compact, S_1 has a limit point in A. Hence S has a limit point in A. If S_2 is infinite, then using similar reasoning, S has a limit point in B. Either way, S has a limit point in $A \cup B$, as desired. □

Problem 2. Prove that $(n^2+5)/(2n^2-n-1) \to 1/2$. You may use the definition of convergence, or you may use any other valid properties of limits from class. Do not use l'Hôpital's rule.

Proof. We first factor out n^2 from the numerator and denomenator to get:

$$\frac{n^2+5}{2n^2-n-1} = \frac{1+\frac{5}{n^2}}{2-\frac{1}{n}-\frac{1}{n^2}}$$

Since $\frac{5}{n^2} \to 0$, the numerator converges to 1. By similar reasoning, the denominator converges to 2. Since the limit of the quotients is the quotient of the limits, the whole fraction converges to $1/2$. □

Problem 3. Suppose that $x_n \to x$ and $y_n \to x$. Prove that the sequence

$$x_1, y_1, x_2, y_2, x_3, y_3, \ldots$$

converges to x too. Use the definition of convergence.

Proof. We need a name for this sequence, let us call it a_n. Now, let $\epsilon > 0$ be given; we must show that eventually every term of a_n lies inside $N_\epsilon(x)$. Since $x_n \to x$, there exists N such that $n > N$ implies $d(x_n, x) < \epsilon$. Since $y_n \to x$, there exists N' such that $n > N$ implies $d(y_n, x) < \epsilon$.

Now, let $N'' = \max\{N, N'\}$. Then for $n > N''$, we have that both $d(x_n, x) < \epsilon$ and $d(y_n, x) < \epsilon$. In other words, eventually every term in the sequence a_n is within ϵ of x. Since ϵ was arbitrary, this implies that a_n converges to x. □

Problem 4. Find the value of the infinite series:

$$1/3 - 1/9 + 1/27 - 1/81 + - \cdots$$

Proof.

$$\begin{aligned} 1/3 - 1/9 + 1/27 - 1/81 + - \cdots &= \frac{1}{3}(1 - 1/3 + 1/9 - 1/27 + - \cdots) \\ &= \frac{1}{3}\sum_0^\infty(-\frac{1}{3})^n \\ &= \frac{1}{3} \cdot \frac{1}{1 - (-\frac{1}{3})} \\ &= \frac{1}{3} \cdot \frac{3}{4} \\ &= \frac{1}{4} \end{aligned}$$

□

Problem 5. Determine whether the following series converges or diverges:

$$\sum(\sqrt{n+1} - \sqrt{n})^n$$

Proof. Using the root test, we must evaluate the limit:

$$\begin{aligned}\lim \sqrt[n]{(\sqrt{n+1}-\sqrt{n})^n} &= \lim \sqrt{n+1}-\sqrt{n} \\ &= \lim \frac{\sqrt{n+1}-\sqrt{n}}{1} \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \\ &= \lim \frac{1}{\sqrt{n+1}+\sqrt{n}} \\ &= 0\end{aligned}$$

Since $0 < 1$, the root test implies that the original series converges. □

Problem 6. Let S be the set of all real numbers in the unit interval $[0, 1]$ whose decimal expansion uses only the digits 4 and 7. Prove that S is compact.

Proof. Let S_n be the set of real numbers in $[0, 1]$ whose first n decimal digits are all either 4 or 7. Then for instance,

$$\begin{aligned}S_1 &= [.4, .5] \cup [.7, .8] \\ S_2 &= [.44, .45] \cup [.47, .48] \cup [.74, .75] \cup [.77, .78]\end{aligned}$$

The pattern continues, and in particular each S_n is a finite union of closed intervals. It follows that $S = \bigcap S_n$ is closed, and since it is obviously bounded, it is compact.

(Note: it is interesting to understand why .5 is in S_1 when its first digit is not a 4 or 7. This is because .5 can also be written as $.4\bar{9}$. Similarly, $.8 = .7\bar{9}$ is in S_1, and $.45 = .44\bar{9}$ is in S_2, and so forth.) □

Problem 7. Determine whether the following series converges or diverges:

$$\sum \frac{1}{(\log n)^{\log n}}$$

Proof. By the condensation test, it suffices to check the following series for convergence.

$$\sum \frac{2^k}{(\log 2^k)^{\log 2^k}} = \sum \frac{2^k}{(k \log 2)^{k \log 2}}$$

We now apply the root test to the latter series, that is, we check the limit

$$\lim \sqrt[k]{\frac{2^k}{(k \log 2)^{k \log 2}}} = \lim \frac{2}{(k \log 2)^{\log 2}} = 0$$

Since $0 < 1$, the root test implies that the condensed series (the one with k) converges. Hence the condensation test implies the original series (the one with n) converges.

(Note: this computation is a bit ugly since $\log 2$ is scattered throughout it. But $\log 2$ is just some constant between 0 and 1, so it is harmless.) □