

7.7 You should know how to compute the definite improper integral  $\int_1^\infty f(x)dx$  because you will need it for the integral test. Remember:

If  $F$  is the antiderivative of  $f$  then  $\int_1^\infty f(x)dx = \left(\lim_{x \rightarrow \infty} F(x)\right) - F(1)$ .

8.1 Compute the limit of a sequence of numbers. Methods include: factor highest powers, rationalize numerator or denominator, squeeze, L'Hôpital's rule, take the log-limit.

Try problems 14, 20, 25, 34

8.2 Divergence test. If  $\lim a_k \neq 0$  then the series  $\sum a_k$  diverges. The converse is *very* false!

Geometric series test. A geometric series is one where each successive term is obtained from the last by multiplying by a fixed real number  $r$ , called the ratio. For example,  $\sum_0^\infty ar^k$ .

- If  $|r| < 1$  then the series converges to  $\frac{a}{1-r}$ .
- If  $|r| \geq 1$  then the series diverges.

Try problems 12, 13, 17, 31, 33

8.3 Integral test. Suppose you have a series  $\sum_1^\infty a_k$  where  $a_k = f(k)$  and  $f(x)$  is a decreasing function. Then:

- If  $\int_1^\infty f(x) < \infty$  then the series converges.
- If  $\int_1^\infty f(x) = \infty$  then the series diverges.

$p$ -test. This can be derived from the integral test. Given the series  $\sum_1^\infty \frac{1}{k^p}$

- If  $p < 1$  then the series converges.
- If  $p \geq 1$  then the series diverges.

8.4 Direct comparison test. Suppose that  $\sum a_k$  is a positive-term series.

- To show that  $\sum a_k$  converges, it suffices to show that  $a_k \leq b_k$  and  $\sum b_k$  converges.
- To show that  $\sum a_k$  diverges, it suffices to show that  $a_k \geq b_k$  and  $\sum b_k$  diverges.

Limit comparison test. Again suppose that  $\sum a_k$  is a positive-term series that you wish to test for convergence. Suppose that  $\sum b_k$  is another positive-term series whose convergence is *known* by some other test. Let

$$L = \lim_{k \rightarrow \infty} \frac{a_k}{b_k}$$

then

- If  $0 < L < \infty$  then  $\sum a_k$  does whatever  $\sum b_k$  does.
- If  $L = 0$  and  $\sum b_k$  converges, then  $\sum a_k$  converges.
- If  $L = \infty$  and  $\sum b_k$  diverges, then  $\sum a_k$  diverges.

8.5 Ratio test. Let  $\sum a_k$  be any series, hopefully one that has exponentials or factorials in it. Let

$$L = \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right|$$

then

- If  $L < 1$  then  $\sum a_k$  converges.
- If  $L > 1$  then  $\sum a_k$  diverges.
- If  $L = 1$  then the test failed, use another.

Root test. Same conclusion, only use  $L = \lim_{k \rightarrow \infty} |a_k|^{1/k}$

8.6 Given a series  $\sum a_k$ , the corresponding absolute series is  $\sum |a_k|$ . If the absolute series converges, the series is called AC.

A positive-term series is always either AC or D.

An alternating series may be AC, CC, or D. If the absolute series diverges, then you must decide between CC or D using the alternating series test or the divergence test.

Alternating series test. Given a series with alternating signs,  $\sum (-1)^k b_k$ , if

- $b_{k+1} \leq b_k$ , ie the  $b_k$  decrease
- $\lim_{k \rightarrow \infty} b_k = 0$

then the series converges (although it may be conditional).

8.7 A power series (centered at zero) is of the form  $\sum a_k x^k$ . Use the ratio test to find its radius of convergence  $R$ . Plug in the endpoints and use other tests to find whether the interval of convergence is either  $(-R, R)$ ,  $(-R, R]$ ,  $[-R, R]$ , or  $[-R, R)$ .

You can also have the series centered at  $x = a$ . Then it looks like  $\sum a_k (x - a)^k$ . The interval of convergence is centered at  $a$  too, so it's either  $(a - R, a + R)$ ,  $(a - R, a + R]$ ,  $[a - R, a + R]$ , or  $[a - R, a + R)$ .

You can differentiate or integrate power series term-by-term.

Try problems 10, 16, 18, 28

8.8 The Maclaurin series for  $f$  is the power series that best approximates  $f$  at  $x = 0$ . It is given by

$$f(x) \sim \sum_0^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

where  $f^{(k)}(x)$  is the  $k^{\text{th}}$  derivative of  $f$  at  $x = 0$ .

You should know the Maclaurin series for  $\frac{1}{1-x}$ ,  $\sin x$ ,  $\cos x$ ,  $e^x$ . Furthermore, many other series can be obtained from these by integrating, differentiating, or substituting. Examples include  $\frac{1}{1+x^2}$ ,  $\ln(1-x)$ ,  $e^{2x}$ ,  $\tan^{-1} x$ .

Try problems 3, 5, 7, 20, 50

Use practice problems from the Chapter 8 Review (page 567 in my book). Test the following for AC, CC, or D: 31, 32, 33, 39, 41, 43, 44, 52, 53, 54, 55