INSTRUCTIONS

- Relax—this exam is no longer and no harder than the second midterm.
- Complete all the parts you can! If you can't get one then continue on anyway!
- An incomplete argument or proof outline is better than no proof.
- Choose only ONE of problems 3 and 4 and check accordingly:
  - ☐ I chose to do problem 3
  - ☐ I chose to do problem 4
- Have a wonderful summer!

1. (a) Let $\phi(x)$ denote the wff that means "x is a perfect square." What subset does $\phi(x)$ define...

   1. in $(\mathbb{N}; +, \cdot)$?

   2. in $(\mathbb{R}; +, \cdot)$?

   3. in $(\mathbb{C}; +, \cdot)$?

   (b) Define "theory" and say what it means for a theory to be "complete."

   (c) If $\mathcal{A}$ is some structure, define $\text{Th}(\mathcal{A})$.

   (d) Suppose that ϕ is a sentence and $\mathcal{A} \vDash \phi$ (*i.e.*, $\mathcal{A}$ satisfies ϕ). Does it necessarily follow that $\text{Th}(\mathcal{A}) \vDash \phi$ (*i.e.*, $\text{Th}(\mathcal{A})$ logically implies ϕ)?

2. A graph G with edge relation $\sim$ is said to be *connected* iff for any two vertices v, w there exists a finite sequence $\{v_i\}$ of vertices such that

$$v \sim v_1 \sim \cdots \sim v_n \sim w$$

Let $\sim$ be the edge relation on $\mathbb{N}$ defined by $n \sim m$ iff $|n - m| = 1$. That is, n, m are joined by an edge iff they are consecutive. Clearly, the graph $(\mathbb{N}; \sim)$ is connected.

(a) Show that $\sim$ is a definable relation on $(\mathbb{N}; +, \cdot, 0, 1, <)$.

(b) Prove that there exists a graph G such that G is elementarily equivalent to $(\mathbb{N}; \sim)$ but such that G is not connected.

(Continued)

(c) Let $\mathcal{C}$ be a class of structures for a fixed language $\mathcal{L}$. Define what it means for $\mathcal{C}$ to be axiomatizable.

(d) Use part (b) to conclude that the class $\mathcal{C}$ of connected graphs is not axiomatizable.

END OF MANDATORY PROBLEMS. Do ONE of the next two problems!

3. (a) State the Vaught test for completeness of a first-order theory.

(b) Let $\mathcal{L} = \{<, P, Q, c\}$ be the language which includes a binary relation $<$, two unary predicates P and Q, and a constant symbol c. Let DLO be the theory of dense linear orders without endpoints and let T be the theory:

$$T = DLO \cup \{\neg\exists x(Px \wedge Qx), \forall x(Px \vee Qx), \forall x((x \leq c) \to Px), \forall x((c < x) \to Qx)\}$$

Prove that T is complete.

4. (a) State the compactness theorem for propositional logic.

(b) Suppose that G is a graph with edge relation $\sim$. We say that G is *bipartite* iff it is possible to partition the vertices $G = P \sqcup Q$ so that every edge adjoins some vertex in P to some vertex in Q. (In other words, no pair of vertices in P are adjacent and no pair of vertices in Q are adjacent.)

Prove that if G is a countable graph and every finite subgraph $G_0 \subset G$ is bipartite, then G is bipartite.

END OF EXAM