1. (a) Let A be a set. Define the following terms.
   - A is uncountable.
   - R is a binary relation on A.
   - R is a partial order on A.
   - R is a linear order on A.
   - R is an equivalence relation on A.

   (b) Let Σ be a set of wffs and let ϕ be a wff. Define the following terms.
   - Σ is satisfiable.
   - Σ is decidable.
   - Σ is complete.
   - $\Sigma \vDash \phi$.
   - ϕ is a tautology.

2. (a) Let $\mathbb{Z}[x]$ denote the set of polynomials in x with integer coeffecients. Prove that $\mathbb{Z}[x] \sim \mathbb{N}$.

(b) Let $\mathbb{Z}[[x]]$ denote the set of power series in x with integer coefficients, *i.e.*, the set of all formal summations $\sum_{i=0}^{\infty} a_i x^i$ where $a_i \in \mathbb{Z}$ for every i. Prove that $\mathbb{Z}[[x]]$ is uncountable.

(c) Let $C_{1/2} \subset \mathbb{Z}[[x]]$ denote the set of power series in x which converge at $x = 1/2$. Is $C_{1/2}$ countable or uncountable? Prove that you are correct.

3. (a) Let S denote the set $\{1 - 1/n : n \in \mathbb{N}, n \neq 0\}$. Decide whether the following pairs of subsets of $\mathbb{Q}$ are isomorphic linear orders. In each case, prove that you are correct.

   - S and $S \cup \{1\}$
   - S and $\mathbb{N}$

   (b) Consider the "lexicographic" ordering of the rational plane $\mathbb{Q}\times\mathbb{Q}$, defined by $(a, b) \prec (a', b')$ iff either:

   - $a < a'$, or
   - $a = a'$ and $b < b'$.

   Prove that $\prec$ is a linear order on $\mathbb{Q} \times \mathbb{Q}$.

   (c) Prove that $(\mathbb{Q} \times \mathbb{Q}, \prec)$ and $(\mathbb{Q}, <)$ are isomorphic linear orders.

4. The binary relation $\leq$ on a set A is called a *preorder* iff

- $\leq$ is reflexive, that is, $a \leq a$ for all $a \in A$, and
- $\leq$ is transitive, that is, if $a \leq b$ and $b \leq c$ then $a \leq c$.

Let $(A, \leq)$ be a preorder. Define a relation $\approx$ on A by $a \approx b$ iff $a \leq b$ and $b \leq a$.

(a) Prove that $\approx$ is an equivalence relation.

(b) Let $[A]$ denote the set of equivalence classes of $\approx$. Define a relation $<$ on $[A]$ by $[a] < [b]$ iff $a \leq b$ and $a \not\approx b$. Prove that $<$ is well-defined and that it is a partial order on $[A]$.

(c) Let $\bar{\mathcal{L}}$ denote the set of all wffs in a countable propositional language. Define the relation $\leq$ on $\bar{\mathcal{L}}$ by $\alpha \leq \beta$ iff the wff $(\alpha \to \beta)$ is a tautology. Prove that $\leq$ is a preorder.

(d) Let $<$ denote the partial order on the set $[\bar{\mathcal{L}}]$ of $\approx$-classes defined in (b). The set $[\bar{\mathcal{L}}]$ has a unique maximal element with respect to this partial order. What is it? Explain.

5. (a) State the Compactness Theorem for propositional logic.

(b) Let A and B be countable sets, and suppose that $F : A \to \mathcal{P}(B)$ is a function which satisfies:

- For each $a \in A$, $F(a)$ is a finite nonempty subset of B.
- For every finite subset $A_0 \subset A$, there exists an injective function $f_0 : A_0 \to B$ such that $f_0(a) \in F(a)$ for all $a \in A_0$.

Use the Compactness Theorem to prove that there exists an injective function $f : A \to B$ such that $f(a) \in F(a)$ for all $a \in A$.