

1. Consider the language $\mathcal{L} = \{R\}$, a single binary relation. Let Σ be the set of sentences:

$$\{\forall x(\neg xRx), \forall x\forall y\forall z(xRy \wedge yRz \rightarrow xRz), \forall x\forall y(x \neq y \rightarrow (xRy \vee yRx))\}$$

Let σ be the sentence:

$$\exists x\forall y(yRx \vee y = x)$$

- (a) What are the models of Σ and what does σ say?
- (b) Define what $\Sigma \models \sigma$ means (in general, not just with this specific Σ, σ).
- (c) Prove that $\Sigma \not\models \sigma$.
- (d) Prove that $\Sigma \not\models \neg\sigma$.

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2. Let $\mathcal{N} = (\mathbb{N}, +, \times, 0, 1, <)$ denote the standard model of arithmetic.
- (a) Show that the set of prime numbers is a definable subset of \mathcal{N} .
- (b) Show that the relation “x and y have exactly the same prime divisors” is definable in \mathcal{N} .
- (c) Define what it means that a structure \mathcal{A} is a nonstandard model of arithmetic.
- (d) Let \mathcal{A} be some nonstandard model of arithmetic. Show that there exists a nonstandard prime number, *i.e.*, an element $a \in A \setminus \mathbb{N}$ whose only divisors are 1 and a .

3. Recall that the partial order $\mathcal{T} = (T; <)$ is a *tree* iff
1. there exists a unique minimal element t_0 (the root), and
 2. for each $t \in T$ the set $\{s \in T : s < t\}$ is finite and linearly ordered.

Let \mathcal{T} be a tree.

- (a) Let $\mathcal{L} = \{<, c\}$ where c is a new constant symbol and let $(T; <, c^T)$ be any \mathcal{L} -structure made out of \mathcal{T} .

Show that for any $n \in \mathbb{N}$, there exists a wff $\phi_n(x)$ such that $(T; <, c^T) \models \phi_n(c)$ holds iff $|\{s \in T : s < c^T\}| \geq n$.

- (b) Let $\Sigma = Th(\mathcal{T})$ (the sentences true in \mathcal{T}). Let $\Sigma' = \Sigma \cup \{\phi_n(c) : n \in \mathbb{N}\}$. Assuming that \mathcal{T} has no empty levels, prove that Σ' is finitely satisfiable.

(continued)

(c) Use the compactness theorem to argue that Σ has an \mathcal{L} -model in which c is interpreted as an “infinite” element (and say what this means, too).

(d) Conclude that the class \mathcal{C} of trees is not axiomatizable.

END OF EXAM