The classification of torsion-free abelian groups up to isomorphism and quasi-isomorphism

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A concise and selective history

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Question

What does it mean for one classification problem to be strictly more complex than another?

Standard Borel spaces

Definition

A standard Borel space is a Polish space X equipped just with its σ-algebra of Borel sets.

Example

$\mathbb{R}, \mathbb{Q}_p, \mathcal{P}(\mathbb{N})$, Borel subsets of these

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The space TFA_n of torsion-free abelian groups of rank n.

This is the standard Borel space consisting of those $A \in \mathcal{P}(\mathbb{Q}^n)$ which are subgroups of $\mathbb{Q}^n$ of rank n.

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Remark

Now, studying the classification problem for torsion-free abelian groups of rank n amounts to studying the isomorphism equivalence relation on TFA_n.

Borel reducibility of equivalence relations

Definition

Let E, F be equivalence relations on standard Borel spaces X, Y. Then E is Borel reducible to F (written $E \leq_B F$) iff there exists a Borel map $f : X \to Y$ satisfying:

$$a \; E \; b \iff f(a) \; F \; f(b)$$

Meaning. . .

- Any set of invariants for F can be used as invariants for E.
- The E-classification problem on X is no harder than the F-classification problem on Y.

Example of a Borel reduction

Torsion-free abelian groups

Definition

Let $\cong_n$ be the isomorphism equivalence relation on the space TFA_n of torsion-free abelian groups of rank n.

Fact

$\cong_n \ \leq_B \ \cong_{n+1}$

Proof.

Use the map $A \mapsto A \oplus \mathbb{Q}$. □

Hjorth's 1998 theorem and Thomas's 2001 theorem

Theorem

The classification problem for torsion-free abelian groups of rank n increases strictly in complexity with the rank n. In symbols:

$$\cong_1 \;<_B\; \cong_2 \;<_B\; \cong_3 \;<_B\; \cdots \;<_B\; \cong_n \;<_B\; \cdots$$

(The first $<_B$ is Hjorth's part.)

Quasi-isomorphism

Definition

Subgroups $A, B \leq \mathbb{Q}^n$ are said to be quasi-isomorphic (written $A \sim_n B$) iff A and B have isomorphic subgroups of finite index.

Thomas found the quasi-isomorphism relation simpler to work with and initially proved:

Theorem (Thomas, 2001)

$$\sim_1 \; <_B \; \sim_2 \; <_B \; \sim_3 \; <_B \; \cdots \; <_B \; \sim_n \; <_B \; \cdots$$

Isomorphism versus quasi-isomorphism

The question

Theorem (Corner)

There exists a torsion-free abelian group A of rank 3 such that

$$A_1 \oplus A_2 \;\cong\; A \;\cong\; B_1 \oplus B_2 \oplus B_3$$

and A_i, B_j are indecomposable!

Theorem (Jónsson)

There is unique decomposition of torsion-free abelian groups in the quasi-isomorphism category.

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Question

Is quasi-isomorphism simpler ($\leq_B$) than isomorphism?

Isomorphism versus quasi-isomorphism

The answer

Answer

Isomorphism and quasi-isomorphism of p-local torsion-free abelian groups of rank n are incomparable, meaning that there is not a Borel reduction either way.

Definition

Let p be a prime. Then $A \leq \mathbb{Q}^n$ is p-local iff it is infinitely q-divisible for every $q \neq p$.

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Isomorphism and quasi-isomorphism of p-local torsion-free abelian groups of rank n are incomparable, meaning that there is not a Borel reduction either way.

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Conjecture

The same is true for isomorphism and quasi-isomorphism on the space of all torsion-free abelian groups of rank n.

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Simon Thomas, Rutgers University

A descriptive view of geometric group theory

Wednesday 1pm (room 1A)