Countable Borel equivalence relations and torsion-free abelian groups

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Standard Borel spaces

Definition

A standard Borel space is a complete separable metric space, but we remember only the algebra of Borel sets.

Examples

- $\mathbb{R}$, $\mathbb{Q}_p$, $\mathcal{P}(\mathbb{N})$, $\mathbb{N}^{\mathbb{N}}$, Borel subsets of these
- Let $\mathcal{L} = \{R_i\}$ be a countable relational language, where R_i is n_i-ary. Then $X_{\mathcal{L}} = \prod \mathcal{P}(\mathbb{N}^{n_i})$ is the class of $\mathcal{L}$-structures on $\mathbb{N}$
- Let σ be a sentence of $\mathcal{L}_{\omega_1,\omega}$. Then $\mathrm{Mod}(\sigma)$ is the subclass of all $\mathcal{M} \in X_{\mathcal{L}}$ such that $\mathcal{M} \models \sigma$

Remark

All uncountable standard Borel spaces are isomorphic.

Classification problems as equivalence relations

Definition (ad hoc)

A concrete classification problem is an equivalence relation E on a standard Borel space X.

Examples

- the isomorphism relation $\cong_\sigma$ on $\mathrm{Mod}(\sigma)$
- eventualy agreement of elements of $\mathbb{N}^\mathbb{N}$
- Turing equivalence on $\mathcal{P}(\mathbb{N})$

Classifiability by countable structures

Definition

An arbitrary concrete classification problem E on X is said to be classifiable by countable structures iff there exists a Borel function $f : X \to \mathrm{Mod}(\sigma)$ satisfying:

$$x \, E \, x' \iff f(x) \cong_\sigma f(x')$$

Definition

The above relationship is abbreviated "E is Borel reducible to $\cong_\sigma$" and written $E \ \leq_B \ \cong_\sigma$.

Fundamental problem

What can be said about the structure of the partial order $\leq_B$?

Extremes

Definition

$\mathrm{Mod}(\sigma)$ is said to be completely classifiable iff $\cong_\sigma \leq_B =_{\mathbb{R}}$, the equality relation on $\mathbb{R}$.

Examples

- Mod(vector spaces over $\mathbb{Q}$)
- Mod(divisible groups)

Definition

$\mathrm{Mod}(\sigma)$ is said to be complete iff $\cong_\tau \leq_B \cong_\sigma$ for every τ.

Examples

- Mod(connected graphs)
- Mod(groups)
- Mod(linear orders)

Essentially countable classes

Definition

$\mathrm{Mod}(\sigma)$ is called essentially countable iff there exists a Borel $B \subset \mathrm{Mod}(\sigma)$ which meets each isomorphism class countably.

Theorem

Suppose that any $\mathcal{M} \in \mathrm{Mod}(\sigma)$ is determined up to isomorphism by some $\bar{a} \in M^n$ and countably many $\mathcal{L}_{\omega_1,\omega}$-formulas over $\bar{a}$. Then $\mathrm{Mod}(\sigma)$ is essentially countable.

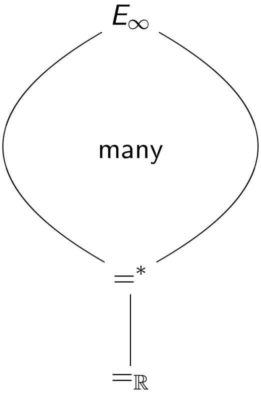
Examples

- Mod(connected locally finite graphs)
- Mod(finitely generated groups)
- Mod(torsion-free abelian groups of finite rank)

Countable Borel equivalence relations

Definition

E is a countable Borel equivalence relation on X iff E is a Borel subset of $X \times X$ and every E-class is countable.



e.g., locally finite graphs, f. g. groups

e.g., torsion-free abelian groups of finite rank

almost equality on 2^ω

equality on $\mathbb{R}$

Torsion-free abelian groups of finite rank

Fact

Any torsion-free abelian group of rank n is isomorphic to a subgroup of $\mathbb{Q}^n$.

Definition

- TFA_n is the standard Borel space of all $A \leq \mathbb{Q}^n$ which span $\mathbb{Q}^n$
- $\cong_n$ is the isomorphism relation on TFA_n

Theorem (Hjorth 1998 and Thomas 2001)

The classification problem for torsion-free abelian groups of rank n increases strictly in complexity with the rank n. In symbols:

$$\cong_1 \ <_B \ \cong_2 \ <_B \ \cong_3 \ <_B \ \cdots \ <_B \ \cong_n \ <_B \ \cdots$$

Quasi-isomorphism

Definition

Subgroups $A, B \leq \mathbb{Q}^n$ are quasi-isomorphic (written $A \sim_n B$) iff A and B have isomorphic subgroups of finite index.

Theorem (Corner)

There exists a torsion-free abelian group A of rank 3 such that

$$A_1 \oplus A_2 \;\cong\; A \;\cong\; B_1 \oplus B_2 \oplus B_3$$

and A_i, B_j are indecomposable!

Theorem (Jónsson)

There is unique decomposition of torsion-free abelian groups in the quasi-isomorphism category.

Question

Which is more complex, isomorphism or quasi-isomorphism?

Isomorphism versus quasi-isomorphism

Theorem ()

Isomorphism and quasi-isomorphism of p-local torsion-free abelian groups of rank n are incomparable, meaning that there is not a Borel reduction either way.

Definition

Let p be a prime. Then $A \leq \mathbb{Q}^n$ is p-local iff it is infinitely q-divisible for every $q \neq p$.