A COMBINATORIAL EXERCISE

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Say that n people are seated sequentially around a circular table. In front of each person is a nametag, with the name of some other person at the table. If the table is rotated, then each person will be sitting in front of a different nametag. Suppose that the arrangement is such that after any rotation of the table, precisely one person is sitting in front of his own nametag. What can we say about n?

We translate the problem into mathematical language. Fix names $0, \ldots, n-1$ for the people at the table. An arrangement of nametags is just a permutation π where $\pi(i)$ is the person seated in front of the nametag for i. Define a map

$$d_\pi(i) = \pi(i) - i \mod n$$

which is in some sense distance between person i and his nametag, according to π.

Now, the condition that after any rotation of the table precisely one person sits in front of his own nametag translates to the following condition on π, which we shall call property (T):

$$d_\pi : n \to n \text{ is injective}$$

This equivalence is easy; $d_\pi(i) = d_\pi(j) = d$ means that rotation by $n-d$ puts both i and j in front of their own nametag.

If n is odd, it is easily seen that there is a π satisfying property (T). Just put the nametag for person i in front of person $2i \mod n$. In other words, let $\pi(i) = 2i \mod n$. Since 2 is a unit modulo n, this map is a permutation. Furthermore, $d_\pi(i) = 2i - i = i$ is clearly injective.

We shall shortly see that the converse also holds. That is, if n is even then there is no arrangement π satisfying property (T).

Lemma. If π is any permutation on n, then $\sum_n d_\pi \equiv 0 \mod n$.

Proof. If σ is a cycle, then the sum telescopes modulo n. Otherwise, decompose σ as a product of disjoint cycles. The sum then divides into a sum over each cycle and is hence again 0. □

Theorem. If π is a permutation on n satisfying property (T), then n is odd.

Proof. By property (T),

$$\sum_n d_\pi = \sum_{i \in n} i = \frac{n(n-1)}{2}$$

By the lemma, this quantity is divisible by n. In other words $(n-1)/2$ is an integer. Hence $n-1$ is even and n is odd. □

The conclusion is that the situation proposed in the first paragraph is possible if and only if n is odd.