SEQUENTIAL TANGENTS

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In the article, Tangent Sequences, World Records, π, and the Meaning of Life: Some Applications of Number Theory to Calculus [2], Ira Rosenholtz wishes to understand the convergence of series of the form

$$\sum_{n=1}^{\infty} \frac{\tan(n)}{n^b}$$

for small natural numbers, b. Unfortunately, this question is a very difficult question to answer, because the tangent function is not bounded for values near $\pi/2$. Rosenholtz immediately simplifies his exploration.

If a series converges, then the sequence composed of the terms of the series must approach zero. In our case, $\lim_{n\to\infty} \frac{\tan(n)}{n^b}$ must equal to zero. The remainder of Rosenholtz's analysis attempts to determine whether this limit equals zero for $b = 1$, 8, and 2.

Continued fractions. It is immediately clear that $\frac{|\tan(n)|}{n^b} < \infty$ for all n, $0 < n < \infty$. This is because $\tan(n)$ is only infinite at odd multiples of $\pi/2$, and there can be no integer n that is an odd multiple of $\pi/2$. However, in this study it is very useful to know which values of n are nearest to an odd multiple of $\pi/2$, and therefore will cause $|\tan(n)|$ to be very large.

Suppose that n is near to the mth multiple of $\pi/2$. Then $n/m \approx \pi/2$. The closer n/m is to $\pi/2$, the closer n is to a multiple of $\pi/2$, the closer $|\tan(n)|$ is to ∞. Thus we would like to find fractions, n/m, that very closely approximate $\pi/2$. Such fractions are called convergents, and can be represented in continued fraction form.

A continued fraction is defined by a sequence of integers, $\{a_n\}$. The a_n should be positive, except that a_0 must be negative if the value of the continued fraction is negative. The value of a continued fraction should be computed as shown:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} \tag{1}$$

For example, the infinite continued fraction defined by $\{1\}_0^\infty$ is equal to the golden ratio, $\frac{\sqrt{5}+1}{2}$.

When $\{a_n\}$ is a finite sequence, this continued fraction is equal to a rational number, p/q. When $\{a_n\}$ is an infinite sequence, this continued fraction converges to an irrational number, called α. More specifically, each continued fraction represented by $\{a_n\}_0^k$ is equal to a rational number, p_k/q_k, and the $\{p_k/q_k\}$ converge to α as $k \to \infty$. The p_k/q_k's are called the convergents of α.

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Conversely, and most helpfully, every rational number p/q can be represented by a finite continued fraction, and every irrational number α can be represented by a unique infinite continued fraction. Conveniently, there is an algorithm to compute the continued fraction for a given rational number. One defines a new sequence, $\{r_n\}$, mutually recursively with $\{a_n\}$ thus:

$$\begin{aligned} r_0 &= \alpha \\ a_n &= [r_n] \\ r_{n+1} &= \frac{1}{r_n - a_n} \end{aligned} \tag{2}$$

But we are interested in the convergents of α, which can also be found recursively in the following way:

$$\begin{aligned} p_0 &= a_0 & q_0 &= 1 \\ p_1 &= a_0 a_1 + 1 & q_1 &= a_1 \\ p_k &= a_k p_{k-1} + p_{k-2} & q_k &= a_k q_{k-1} + q_{k-2} \end{aligned} \tag{3}$$

The equations (3) are simple to prove inductively. It is difficult, however, to prove that the sequence defined by equations (2) converge to α. The proof relies on the equations (3), and can be found in [1].

It is important to note two things. First, from the above formulas, it is clear that the sequences $\{p_k\}$ and $\{q_k\}$ are increasing. Second, a given convergent, p/q, to α is the best rational approximation to α among all possible fractions with denominator less than or equal to q. In upcoming sections, Rosenholtz sets n/m equal to the $\{p_k/q_k\}$ of $\pi/2$. Thus, when $n = p_k$, and $m = q_k$ is odd, $|\tan(n)|$ is large. To calculate a_n for large values of n, we need to know many digits of the irrational α already. Fortunately, billions of digits of $\pi/2$ are known!

Convergence for $b = 1$. Rosenholtz begins with the case for $\frac{\tan(n)}{n}$. That is, we want to know if this sequence approaches 0 as $n \to 0$. The most interesting thing, and perhaps the best selling point of this paper, is that the outcome is not obvious (at least not until one looks at the structure of this document).

In fact, Rosenholtz finds that $\frac{\tan(n)}{n}$ does not converge at all. I will explain his proof in detail.

The proof that $\lim_{n\to\infty} \frac{\tan(n)}{n}$ does not exist comes in two parts. First, he shows that if the limit were to exist, it must be zero. Second, he shows that the limit cannot be zero. The first statement he proves in two ways; I will discuss the method that does not involve drawing a complicated picture.

For the first step, it is enough to find a subsequence, $\{n_k\}$, for which $\lim_{n\to\infty} \frac{\tan(n_k)}{n_k} = 0$. In choosing this sequence, we would like for $|\tan(n_k)|$ to be small, perhaps bounded by a constant, K, so that

$$\frac{|\tan(n_k)|}{n_k} \leq \frac{K}{n_k} \to 0 \tag{4}$$

To find such a sequence we ask the question, for what values n is $|\tan(n)|$ small? Well, $\tan(x) = 0$ for $x = \pi k$. Unfortunately, numbers of the form πk are not integers, but we can still find a sequence of numbers that are near enough to multiples of π. Rather arbitrarily, the author chooses the greatest integer function, that is, $n_k = [\pi k]$. Thus, each n_k is no more than 1 away from a multiple of π.

Now, the tangent function is one-to-one on intervals that stay away from odd multiples of $\pi/2$. If $n_k \in [\pi k - 1, \pi k + 1]$, then

$$\begin{aligned} \tan(n_k) &\in [\tan(\pi k - 1), \tan(\pi k + 1)] \\ &= [\tan(-1), \tan(1)] \\ &= [-\tan(1), \tan(1)] \end{aligned}$$

Therefore, $|\tan(n_k)| \leq \tan(1)$, and so we set $K = \tan(1)$ in (4).

So far, we have shown that if a limit of the tangent sequence exists, it must be zero. The second step is to show that the limit cannot be zero. Similarly, it is enough to find a subsequence, $\{n_k\}$, for which $|\tan(n_k)|/n_k$ stays *away* from 0. This time we ask the question, for what values n is $|\tan(n)|$ large? Well, $\tan(x) = \infty$ for $x = k\pi/2$, k odd. Again, there are no integers of this form. But we already know from the last section a sequence of numbers for which $|\tan(n)|$ is large! If $\{p_k/q_k\}$ are the convergents of $\pi/2$, then we know that $|\tan(p_k)|$ is large whenever q_k is odd.

At this point, it is necessary to know that q_k is odd quite often. The following fact is relevant. For the convergents p_k/q_k of α,

$$p_k q_{k+1} - p_{k+1} q_k = (-1)^k \tag{5}$$

This is very easy to prove inductively, and I will skip it. In either case, this fact implies that one of $p_k q_{k+1}$ and $p_{k+1} q_k$ is odd, for any k. If it is the former, then q_{k+1} is odd. If it is the later, then q_k is odd. Therefore, at least half of the q_k are odd.

We now let $\{n_k\}$ equal the infinite subset of $\{p_k\}$ for which q_k is odd, and $q_k > 2$. Let $\{m_k\}$ be these $\{q_k\}$. Thus, we have a sequence of infinitely many integers for which $|\tan(n)|$ is pretty large. The next few steps may seem a little bit odd, but when put together they will show that the $\frac{|\tan(n_k)|}{n_k}$ are bounded below by some number greater than zero.

First, we use another nice fact about the convergents of continued fractions. Suppose $\{p_k/q_k\}$ are the convergents of α. Then, $|p_k/q_k - \alpha| < 1/q_k^2$. Again, I will refer the reader to [1] for the proof. Applying this to our case, we have $|n_k/m_k - \pi/2| < 1/m_k^2$. Multiplying both sides by m_k yields that

$$|n_k - m_k\pi/2| < 1/m_k \tag{6}$$

But since we made sure that $m_k > 2$, we have $1/m_k < 1/2 < \pi/4$. Therefore, $|n_k - m_k\pi/2| < \pi/4$. In other words,

$$n_k \text{ differs from an odd multiple of } \pi/2 \text{ by less than } \pi/4. \tag{7}$$

Next, we will use the mean value theorem! The MVT states that if f(x) is differentiable on $[a, b]$ (more than needed), and $c \in (a, b)$, then

$$f(b) - f(a) = (b - a)f'(c)$$

In this case, we will apply the MVT to the function $\cot(n)$, on the interval $I = [n_k, m_k\pi/2]$. By (7), n_k stays near $m_k\pi/2$, and so n_k stays *away* from multiples of π in I. Thus, $\cot(x)$ is differentiable on I. Note also that since $\cot(m_k\pi/2)$, for odd m, is equal to zero. We can thus write $\cot(n_k)$ as $\cot(n_k) - \cot(m_k\pi/2)$. Plugging this all into the MVT yields

$$\cot(n_k) - \cot(m_k\pi/2) = \cot(n_k) = (n_k - m_k\pi/2)(-\csc^2(c))$$

Taking the reciprocal of several sides of this equation results in

$$\tan(n_k) = \frac{1}{(n_k - m_k\pi/2)(-\csc^2(c))} \tag{8}$$

Now, there are a couple of estimates to be made. First, by (7), we know that n_k stays away from multiples of π on I. Therefore, we can bound the cosecant function:

$$|\csc(n_k)| \le M \text{ on } I. \tag{9}$$

Also, n_k/m_k is equal to some p_k/q_k, which is a convergent of the continued fraction for $\pi/2$. Therefore, since the $\{n_k/m_k\}$ converge to a number ($\alpha = \pi/2$), they must be bounded. Let's say

$$\left|\frac{n_k}{m_k}\right| \le A \tag{10}$$

Now we have all that we needed. Combining (6), (8), (9), and (10), we can say:

$$\begin{aligned} |\tan(n_k)| &\ge \frac{1}{1/m_k \times M^2} \\ \implies \frac{|\tan(n_k)|}{n_k} &\ge \frac{1}{n_k/m_k \times M^2} \\ &\ge \frac{1}{A \times M^2} \\ &> 0 \end{aligned} \tag{11}$$

In summary, Rosenholtz has found two subsequences of $\{n\}$—one for which $\tan(n)/n \to 0$, and another for which $|\tan(n)|/n$ is bounded away from 0. Therefore, the sequence $\tan(n)/n$ does not converge to a limit.

Convergence for $b = 8$. Rosenholtz contrasts the results of the last section with the case for $\frac{\tan(n)}{n^8}$. He finds that in this case, the sequence converges to 0. To prove this, the author shows that for large n, $|\tan(n)/n^8|$ is bounded by an expression whose terms go to zero.

We wish to show that $\tan(n)/n^8 \to 0$. So, for each n we let m be the integer such that $|n - m\pi/2| < \pi/4$. That is, m is the multiple of $\pi/2$ that n is closest to. As before, when m is even, n is near a multiple of π, and stays away from odd multiples of $\pi/2$. Let $\{n_j\}$ be those n for which the corresponding m is even. We know that $|\tan(n_j)|$ stays away from the tangent's bad points, and so is bounded. Thus $|\tan(n_j)| < M$, and so $\frac{|\tan(n_j)|}{n_j^8} < \frac{M}{n_j^8} \to 0$.

On the other hand, if m is odd, we will need to use the mean value theorem as before. Before we do this, however, we must to provide the three inequalities, (13)–(15). For the first, we begin with the following theorem by M. Hata:

$$\exists S > 0, \text{ such that } s > S \implies |r/s - \pi| > \frac{1}{s^{8.02}} \tag{12}$$

Now, we rearrange (12) by setting $r = 2n$, and $s = m$.

$$\begin{aligned} &|r/s - \pi| = |(2/m)n - \pi| > \frac{1}{m^{8.02}} \\ \implies \quad &|n - m\pi/2| > \frac{1}{2m^{7.02}} \end{aligned} \tag{13}$$

Note have now assumed that n is sufficiently large. This does not affect the power of our proof, because finitely many terms in a sequence do not affect its convergence.

Now we move on to our second inequality. We know from (7) that for m odd, n stays near $m\pi/2$, and *away* from multiples of π. Let n_k be those n for which the corresponding m_k is odd. Then, for some $c \in (n_k, m_k\pi/2)$,

$$|\csc(c)| = \frac{1}{\sin(c)} > 1 \tag{14}$$

While I (following Rosenholtz) have written c as a single scalar variable, it should be noted that c will of course be different for each k.

Thirdly, recall that the m_k are chosen so that $|n_k - m_k\pi/2| \leq \pi/4$. We can see from this requirement that the m must increasse with the n. Dividing both sides of this equation by m yields the relationship $|n_k/m_k - \pi/2| \leq \pi/4m_k \to 0$ as $k \to \infty$. So, as k gets large, n_k/m_k becomes close to $\pi/2$ (as we should expect—this shows that n_k/m_k is a convergent of $\pi/2$), which is greater than 1. Therefore,

$$n_k \geq m_k \tag{15}$$

Finally, we put (13), (14), and (15) together with (8) from the previous section.

$$\begin{aligned} |\tan(n_k)| &= \frac{1}{|n_k - m_k\pi/2|} \times \frac{1}{|\csc^2(c)|} \\ &< 2m_k^{7.02} \times 1 \\ &< 2n_k^{7.02} \\ \implies \frac{|\tan(n_k)|}{n_k^8} &< \frac{2}{n_k^{0.98}} \\ &\to 0 \end{aligned} \tag{16}$$

In summary, Rosenholtz finds that for both even and odd natural numbers, $\{n\}$, the sequence with components $\frac{\tan(n)}{n^8}$ converges to 0. It is important to note that Rosenholtz chose the value $b = 8$ because that is the closest approximation that could be made in Hata's result, (12).

Convergence for $b = 2$. The author has saved up the most difficult (and the most fun) of the three cases for last. I will not explain the intermediate steps of proofs in this section in as great detail as I did in the last two sections.

Without a more powerful version of Hata's result from the previous section, Rosenholtz knows of no way to prove that $\frac{\tan(n)}{n^2}$ approaches 0. So, he uses the previously introduced machinery to find numerical bounds for sequence at very large n.

To simplify the question, we are asked to notice that if $\tan(n)/n^2$ approaches some number $M > 0$, then

$$\frac{\tan(n)}{n} = \frac{\tan(n)}{n^2} \times n \to M \times n = \infty$$

Thus, if $\tan(n)/n^2$ does not approach 0, then the $\tan(n)/n$ must be unbounded. The unboundedness of a sequence can be characterized (somewhat loosely) by the fact that such sequences must continue to produce very large values, even as n gets very large. So, if $\tan(n)/n$ is unbounded, it must produce values that are larger than all of the previous values an infinite number of times.

Rosenholtz is thus looking for records. Suppose we have a sequence $\{a_k\}$. He defines a_k *records* to be those numbers K such that

$$\forall(k < K),\ a_k < a_K \tag{17}$$

In words, if $\tan(n)/n$ has an infinite number of records, then it is unbounded. One can imagine where these records might be found. We know that $\tan(n)$ is large for n near $m\pi/2$, if m is odd. Thus, $\tan(n)$ is large if n/m is equal to some convergent p_k/q_k of the continued fraction for $\pi/2$, and m is odd.

Rosenholtz spends quite some time proving the rather obvious result that each of the $\tan(n)/n$ records, $\{K_k\}$, is in fact equal to some p_k. The proof is not straightforward, but neither is it complex. The difficulty comes in showing that when n/m is not a reduced fraction, then n is not a record.

Rather than use the mean value theorem, as in the above sections, the author uses the Laurent expansion of the cotangent function. Why would the expansion of the cotangent function help in estimating tangents? Well, suppose that m is an odd number, and that it equals $2j + 1$. Then the $\cot(n - m\pi/2)$ becomes $\cot(n - \pi/2 - \pi j)$. But the cotangent function is π periodic, and j is an integer, so this is exactly the same as $\cot(n - \pi/2)$. Now,

$$\begin{aligned}
\cot(n - m\pi/2) &= \cot(n - \pi/2) \\
&= \frac{\cos(n - \pi/2)}{\sin(n - \pi/2} \\
&= \frac{\cos(n)\cos(\pi/2) + \sin(n)\sin(\pi/2)}{\sin(n)\cos(\pi/2) - \cos(n)\sin(\pi/2)} \\
&= -\frac{\sin(n)}{\cos(n)} \\
&= -\tan(n)
\end{aligned}$$

But we already know a lot about estimating $|n - m\pi/2|$, so writing out the series for $\cot(n - m\pi/2)$ gives Rosenholtz quite a bit of information about $\tan(n)$, and even $\tan(n)/n$. He uses this tool to produce several key inequalities involving $|\tan(n)|$.

Armed with the fact that $\tan(n)/n$ records are all p_k's, Rosenholtz uses Mathematica to find all such records less than $n = 10^{8,000,000}$. The last record that he found is on the order of $10^{600,000}$. He then uses the following relation to carry over these results to bounding $|\tan(n)|/n^2$. Suppose that K_k is a $\tan(n)/n$ record (recall that records are values of n, not of a_n). Then for all n with $K_k \leq n < K_{k+1}$,

$$\begin{aligned}
& \tfrac{|\tan(n)|}{n} \leq \tfrac{|\tan(K_k)|}{K_k} && \text{(because } K_k \text{ is the record)} \\
\Longrightarrow\ & \tfrac{|\tan(n)|}{n^2} \leq \tfrac{|\tan(K_k)|}{K_k^2} && \text{(because } n > K_k\text{)}
\end{aligned}$$

In this way, Rosenholtz is able to bound $\frac{|\tan(n)|}{n^2}$ by about $1.92 \times 10^{-576,444}$ for all $n \in [5.37 \times 10^{576,450}, 10^{8,000,000}]$. Thus, the best that Rosenholtz can say is that as far as modern hardware is concerned, the $\frac{\tan(n)}{n^2}$ series stays very small for a very long time.

Words $\in$ article. In all, I found this article fun to read, and I found the results exciting. It is clear that the author also had fun writing it—he certainly seemed to enjoy using Mathematica, and making large and small lettering with his typesetting program. It also must have brought him great joy to use the words "theorem" and "proof" inside double quotes.

I have only a few minor difficulties with this paper. First, and most trivially, there is an error in one of the theorems in the $b = 2$ section. The author wrote $m\pi$ when he intended to write $m\pi/2$. This author hopes that he is not held to the same level of scrutiny.

Secondly, the author is often unclear in his use of subscripts. It is difficult for the reader to distinguish those times when he is referring to specific elements of a sequence, and those times when he is referring to general terms in a sequence. I have attempted to clarify those circumstances in my analyses, above.

REFERENCES

[1] Rosen, Kenneth H. *Elementary Number Theory and its Applications.* Addison Wesley: 1999.

[2] Rosenholtz, Ira. "Tangent Sequences, World Records, π, and the Meaning of Life: Some Applications of Number Theory to Calculus." *Mathematics Magazine*, Vol. 72, No. 5, December 1999.