INTEGRALS

Here is a list of the 8 different kinds of integrals you should now be familiar with:

- (Calc 1) Single integral: $\int_a^b f(x)\,dx$
- (Ch 15) Double integral: $\iint_D f(x,y)\,dA$
- (Ch 15) Triple integral: $\iiint_{\mathcal{W}} f(x,y,z)\,dV$
- (Ch 15) Iterated integrals, eg: $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$, $\int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\,dx\,dy$,

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

- (16.2) Line integral of scalar field f on $\mathbb{R}^n$ along a smooth curve $\mathcal{C}$ in $\mathbb{R}^n$: $\int_{\mathcal{C}} f\,ds$
- (16.3) Line integral of a vector field $\vec{F}$ on $\mathbb{R}^n$ along a smooth curve $\mathcal{C}$ in $\mathbb{R}^n$: $\int_{\mathcal{C}} \vec{F}\cdot d\vec{s}$
- (16.4) Surface integral of a scalar field f on $\mathbb{R}^n$ over a smooth surface $\mathcal{S}$ in $\mathbb{R}^n$: $\iint_{\mathcal{S}} f\,dS$
- (16.5) Surface integral of a vector field $\vec{F}$ on $\mathbb{R}^n$ over a smooth surface $\mathcal{S}$ in $\mathbb{R}^n$: $\iint_{\mathcal{S}} \vec{F}\cdot d\vec{S}$

Single, double, and triple integrals, together with line and surface integrals of scalar fields, are each defined as limits of appropriate Riemann sums. Double and triple integrals can be converted to iterated integrals provided the region over which the integral is taken is sufficiently "simple." Line and surface integrals of vector fields are defined in terms of line and surface integrals of scalar fields; specifically, by definition

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{s} = \int_{\mathcal{C}} (\vec{F}\cdot\vec{T})\,ds \quad \text{and} \quad \iint_{\mathcal{S}} \vec{F}\cdot d\vec{S} = \iint_{\mathcal{S}} (\vec{F}\cdot\vec{e}_n)\,dS,$$

where $\vec{T}$ is the unit tangent vector along $\mathcal{C}$ and $\vec{e}_n$ is the unit normal vector across $\mathcal{S}$. If $f = 1$ in the line or surface integral of a scalar field, then we recover the integrals for arc length and surface area, respectively.

Integration Formulas

- $$\int_{\mathcal{C}} f\,ds = \int_a^b f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$$

- $$\int_{\mathcal{C}} \vec{F}\cdot d\vec{s} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

- $$\iint_{\mathcal{S}} f(x,y,z)\,dS = \iint_D f(\Phi(u,v))\,\|\vec{n}(u,v)\|\,dA = \iint_D f(x,y,g(x,y))\sqrt{1+g_x^2+g_y^2}\,dA$$

- $$\iint_{\mathcal{S}} \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\Phi(u,v))\cdot\vec{n}(u,v)\,dA = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

Fundamental Theorems of Calculus

- Fundamental Theorem of Calculus for single integrals: $\int_a^b F'(x)\,dx = F(b) - F(a)$
- Fundamental Theorem of Calculus for line integrals: $\int_{\mathcal{C}} \nabla f\cdot d\vec{s} = f(\vec{r}(b)) - f(\vec{r}(a))$
- Green's Theorem: $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\mathcal{C}} P\,dx + Q\,dy$
- Stokes' Theorem: $\iint_{\mathcal{S}} \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_{\mathcal{C}} \vec{F}\cdot d\vec{s}$
- Divergence Theorem: $\iiint_{\mathcal{W}} \operatorname{div}\vec{F}\,dV = \iint_{\mathcal{S}} \vec{F}\cdot d\vec{S}$