SOLUTIONS FOR WORKSHOP 1

(1) Letting s be the arc length parameter, we first use the arc length function to find the equation relating s and t. We have

$$s = s(t) = \int_0^t ||\vec{r}\,'(u)||\, du = \int_0^t \left\| \left\langle \frac{-4u}{(u^2+1)^2}, \frac{-2u^2+2}{(u^2+1)^2} \right\rangle \right\| \, du$$

$$= \int_0^t \sqrt{\frac{16u^2}{(u^2+1)^4} + \frac{4u^4-8u^2+4}{(u^2+1)^4}}\, du = \int_0^t \sqrt{\frac{4(u^2+1)^2}{(u^2+1)^4}}\, du = \int_0^t \frac{2}{(u^2+1)}\, du.$$

Hence

$$s = 2\arctan t, \quad t = \tan(s/2).$$

Reparameterizing with respect to arc length, we obtain

$$\vec{r}_1(s) = \vec{r}(\tan(s/2)) = \left\langle \frac{2}{\tan^2(s/2)+1} - 1, \frac{2\tan(s/2)}{\tan^2(s/2)+1} \right\rangle, \quad 0 \le s < \pi.$$

Since $\tan^2\theta + 1 = \sec^2\theta$, this simplifies as follows:

$$\vec{r}_1(s) = \left\langle \frac{2}{\sec^2(s/2)} - 1, \frac{2\tan(s/2)}{\sec^2(s/2)} \right\rangle = \langle 2\cos^2(s/2) - 1, 2\tan(s/2)\cos^2(s/2) \rangle$$

$$= \langle \cos s, 2\sin(s/2)\cos(s/2) \rangle = \langle \cos s, \sin s \rangle, \quad 0 \le s < \pi.$$

Therefore, the path traced out by $\vec{r}(t)$, $0 \le t < \infty$, is simply the upper half unit circle centered at the origin. From $t = 0$ to $t = 1$, $\vec{r}(t)$ traces out that part of the unit circle lying in the first quadrant, and then from $t = 1$ to $t = \infty$, $\vec{r}(t)$ traces out that part of the unit circle lying in the second quadrant. Hence as t grows, the progress of $\vec{r}(t)$ along the upper half unit circle becomes slower and slower, so that $\vec{r}(t)$ never actually reaches the point $(-1, 0)$ on the unit circle, taking infinitely long to get there.

(2) Let $ax + by + cz = d$ be an arbitrary plane in $\mathbb{R}^3$, and (x_0, y_0, z_0) an arbitrary point in $\mathbb{R}^3$. We write

$$d_0 = ax_0 + by_0 + cz_0, \quad \text{and} \quad \vec{n} = \langle a, b, c \rangle.$$

The distance from the point to the plane is the length of the line segment connecting the two and perpendicular to the plane. Thus we start by finding an equation of the line through the point and normal to the plane. This line has equation

$$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle.$$

Next we find the intersection of this line with the plane by plugging the line's parametric equations into the equation of the plane. This gives us

$$a(x_0 + at) + b(y_0 + bt) + c(z_0 + ct) = d,$$

and solving for t gives

$$t = \frac{d - d_0}{||\vec{n}||}.$$

If we plug this value for t into $\vec{r}$, we will obtain the point where the line $\vec{r}(t)$ intersects the plane. The distance between the point and the plane is then the length of the line segment connecting (x_0, y_0, z_0) to this point, $\vec{r}\left(\frac{d-d_0}{||\vec{n}||}\right)$. Since

$$\vec{r}\left(\frac{d-d_0}{||\vec{n}||^2}\right) = \left\langle x_0 + a\frac{d-d_0}{||\vec{n}||^2}, y_0 + b\frac{d-d_0}{||\vec{n}||^2}, z_0 + c\frac{d-d_0}{||\vec{n}||}\right\rangle,$$

this distance will be

$$\left\|\left\langle a\frac{d-d_0}{||\vec{n}||^2}, b\frac{d-d_0}{||\vec{n}||^2}, c\frac{d-d_0}{||\vec{n}||^2}\right\rangle\right\| = \sqrt{(a^2+b^2+c^2)\left(\frac{d-d_0}{||\vec{n}||^2}\right)^2}$$

$$= ||\vec{n}||\left(\frac{|d-d_0|}{||\vec{n}||^2}\right) = \frac{|d-d_0|}{||\vec{n}||}.$$

Hence the distance between the plane $ax+by+cz=d$ and the point (x_0, y_0, z_0) is

$$\frac{|d-ax_0-by_0-cz_0|}{\sqrt{a^2+b^2+c^2}},$$

and so the distance between the plane $3x-4y+12z=21$ and the point $(2,-1,2)$ is

$$\frac{|21-2(3)+4(-1)-12(2)|}{\sqrt{3^2+(-4)^2+12^2}} = \frac{13}{13} = 1.$$

(3) Let $P=(1,-1,-2)$, $Q=(-2,2,2)$, $R=(1,5,6)$, and $S=(4,2,2)$. Then

$$\overrightarrow{PQ} = \langle -3,3,4\rangle, \quad \text{so } ||\overrightarrow{PQ}|| = \sqrt{34};$$
$$\overrightarrow{PR} = \langle 0,6,8\rangle, \quad \text{so } ||\overrightarrow{PR}|| = 10; \text{ and}$$
$$\overrightarrow{PS} = \langle 3,3,4\rangle, \quad \text{so } ||\overrightarrow{PS}|| = \sqrt{34}.$$

This implies that P and R are opposite vertices, and Q and S are opposite vertices. Hence the center of the ellipse will be the midpoint of the line segment from P to R, so

$$\text{center } = (1,2,2).$$

The distance from P to the center and from Q to the center are, respectively, 5 and 3. Thus we have

length of major axis: 10

length of minor axis: 6.

According to the formula on page 657 of the textbook, the foci will be a distance of $\sqrt{5^2-3^2}=4$ away from the center, along the major axis. Hence the foci are located at:

$$P + \frac{1}{10}\overrightarrow{PR} = \left(1, -\frac{2}{5}, -\frac{6}{5}\right)$$
$$P + \frac{9}{10}\overrightarrow{PR} = \left(1, \frac{22}{5}, \frac{26}{5}\right).$$

Finally, to parameterize the ellipse,[1] we first find an equation for the plane in which the ellipse lies. The cross product

$$\overrightarrow{PR} \times \overrightarrow{QS} = \langle 0,48,-36\rangle$$

gives a normal vector $\vec{n} = \langle 0,4,-3\rangle$ for this plane. Hence (using the point P),

$$0(x-1)+4(y+1)-3(z+2)=0, \quad \text{or} \quad 4y-3z=2,$$

[1] This is quite difficult; needless to say, you will not be expected to do anything like this on an exam!

is an equation for this plane. Notice that it is orthogonal to the yz-plane, making it somewhat easier to visualize. (In fact, we can now for the first time visualize what the ellipse actually looks like in $\mathbb{R}^3$). Since the plane containing the ellipse is perpendicular to the yz-plane, we will consider the projections of the ellipse onto the xy- and xz-planes to help in finding a parameterization.

First, notice that the projection of the ellipse onto the yz-plane is the circle of radius 3 with center $(x, y) = (1, 2)$, and the projection of the ellipse onto the xz-plane is the ellipse with semimajor axis 4 in the z-direction, semiminor axis 3 in the x-direction, and center $(x, z) = (1, 2)$. (It is helpful at this point to draw these projections). We must now decide how to parameterize the ellipse. From the pictures of the projections, it seems reasonable to parameterize the ellipse by beginning at the point $P = (1, -1, -2)$ and traveling around the ellipse in the direction $P \to S \to R \to Q \to P$, traversing the ellipse once from P back to P as t varies from 0 to 2π.

Therefore, turning our attention to the projection of the ellipse onto the xy-plane, we should parameterize the circle of radius 3 centered at $(x, y) = (1, 2)$ by:

$$\vec{r}_1(t) = \langle x(t), y(t)\rangle = \langle 1 + 3\sin t, 2 + 3\sin(t + 3\pi/2)\rangle, \quad 0 \le t < 2\pi.$$

If we parameterize $x = x(t)$ in this way, then the projection of the ellipse onto the xy-plane should be parameterized by:

$$\vec{r}_2(t) = \langle x(t), z(t)\rangle = \langle 1 + 3\sin t, 2 + 4\sin(t + 3\pi/2)\rangle, \quad 0 \le t < 2\pi.$$

Putting these together, we obtain the final parameterization

$$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle = \langle 1 + 3\sin t, 2 + 3\sin(t + 3\pi/2), 2 + 4\sin(t + 3\pi/2)\rangle, \quad 0 \le t < 2\pi.$$

Plot it in Maple to check that it works!

(4) The Mean Value Theorem does *not* extend to vector-valued functions. For instance, let

$$\vec{r}(t) = \langle \cos t, \sin t, t\rangle, \quad 0 \le t \le 2\pi.$$

Then $\vec{r} : \mathbb{R} \to \mathbb{R}^3$ is continuous on $[0, 2\pi]$ and differentiable on $(0, 2\pi)$, and

$$\vec{r}(2\pi) - \vec{r}(0) = \langle 1, 0, 2\pi\rangle - \langle 1, 0, 0\rangle = \langle 0, 0, 2\pi\rangle.$$

However, the derivative $\vec{r}\,'(t) = \langle -\sin t, \cos t, 1\rangle$ of $\vec{r}(t)$ never points in the direction $\langle 0, 0, 1\rangle$, and so there is no point $c \in (0, 2\pi)$ such that

$$2\pi\vec{r}\,'(c) = \langle 0, 0, 2\pi\rangle.$$