BERNOULLI ACTIONS OF LOW RANK LATTICES AND COUNTABLE BOREL EQUIVALENCE RELATIONS

SCOTT SCHNEIDER

Abstract. Using Zimmer's cocycle superrigidity theorems [27], we obtain Borel nonreducibility results for orbit equivalence relations arising from Bernoulli actions of suitably chosen low-rank lattices in real and p-adic Lie groups. In particular, for p a prime, let E_p (respectively F_p) denote the orbit equivalence relation arising from any nontrivial Bernoulli action of $PSL_2(\mathbb{Z}[\sqrt{p}])$ (respectively $PSL_2(\mathbb{Z}[\frac{1}{p}])$). Then for $p \neq q$, each of the pairs E_p, E_q and F_p, F_q is incomparable with respect to Borel reducibility.

1. Introduction

Any countable group G acts on its power set 2^G by Bernoulli shifts: $(g_0 \cdot x)(g) = x(g_0^{-1}g)$. The product probability measure μ on 2^G is invariant under this action, and the free part

$$(2)^G = \{x \in 2^G \mid g \cdot x \neq x \text{ for all } 1 \neq g \in G\}$$

has μ-measure 1. Let E_G denote the orbit equivalence relation arising from $G \curvearrowright (2)^G$.

It is a consequence of Popa's superrigidity theorem [19] that if the countable group G has no nontrivial finite normal subgroups and contains an infinite normal Kazhdan subgroup, then for *any* countable group H, $E_G \leq_B E_H$ iff G embeds into H. This remarkable fact was used by Thomas in [24] to show that the isomorphism relation on torsion-free abelian groups of finite rank is not countable universal, and that not every countable Borel equivalence relation is Borel bireducible with an orbit equivalence relation arising from a free Borel action of a countable group.

However, even the most general form of Popa's theorem (see [20]) cannot be applied to subgroups of $SL_2(\mathbb{C})$. (This is due to the fact that infinite subgroups of $SL_2(\mathbb{C})$ have amenable centralizers). However, for p a prime, the groups

$$\Gamma_p = PSL_2(\mathbb{Z}[\sqrt{p}]) \quad \text{and} \quad \Delta_p = PSL_2(\mathbb{Z}[\tfrac{1}{p}])$$

may be realized as irreducible lattices in the higher rank connected Lie groups

$$PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R}) \quad \text{and} \quad PSL_2(\mathbb{R}) \times PSL_2(\mathbb{Q}_p),$$

respectively, and hence Zimmer's superrigidity theorems [27, 5.2.5 and 10.1.6] are relevant. In this paper, we will use Zimmer superrigidity together with the techniques of [23] and [21] to show that for $p \neq q$, each of the pairs E_{Γ_p}, E_{Γ_q} and E_{Δ_p}, E_{Δ_q} is incomparable with respect to Borel reducibility.

In fact, let $S = \{p_1, \ldots, p_s\}$ be any finite, nonempty set of primes, let $\mathcal{O}_S$ be the ring of integers inside the number field $\mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_s})$, and define

$$\Gamma_S = PSL_2(\mathcal{O}_S), \qquad \Delta_S = PSL_2(\mathbb{Z}[\tfrac{1}{p_1}, \ldots, \tfrac{1}{p_s}]).$$

Let (Y, ν) be any nontrivial standard Borel probability space, i.e., a (possibly countable or finite) standard Borel space Y together with a Borel probability measure ν on Y that does not concentrate on a point. Then the subset X_S of the product space Y^{Γ_S} (Y^{Δ_S}) on which Γ_S (Δ_S) acts freely is invariant and conull. Let $E_{\Gamma_S}^{X_S}$ $(E_{\Delta_S}^{X_S})$ be the orbit equivalence relation arising from the Bernoulli action of Γ_S (Δ_S) on X_S. In this paper we will show that if T is any other finite nonempty set of primes, then $E_{\Gamma_S}^{X_S} \leq_B E_{\Gamma_T}^{X_T}$ iff $S \subseteq T$, and $E_{\Delta_S}^{X_S} \leq_B E_{\Delta_T}^{X_T}$ iff $S \subseteq T$.

Our proofs of these results will rely heavily upon the techniques used in Thomas [23] and Schneider [21], which in turn draw upon ideas that first appeared in Adams [1]. In fact, our proofs of Theorems 3.1 and 3.2 resemble the proofs of Theorems [21, 3.1] and [23, 3.5] closely enough that in our arguments below we will concentrate primarily on the points of difference from them, and refer the reader to [23] and [21] for many details.

It will be useful, therefore, to recall some of the main ingredients appearing in the proofs of [23, 3.5] and [21, 3.1]. In addition to Zimmer superrigidity, these include:

- the E_0-ergodicity of orbit equivalence relations arising from suitable actions of groups with Property (τ);
- a measure classification theorem for measures on homogeneous spaces of real and p-adic Lie groups;

- various computational techniques involving cocycles and induced spaces that go back to Adams-Kechris [2] and Adams [1].

We will now say a few words about these ingredients, and about the organization of this paper generally.

In [23] and [21], the fact that the groups Γ_S and Δ_S have Property (τ) played a crucial role in setting up applications of Zimmer's cocycle superrigidity theorems to actions of the ambient Lie groups in which they are lattices. In this paper, however, we shall be able to verify directly that Bernoulli actions of Γ_S and Δ_S are E_0-ergodic, and hence we will not require the technology developed in Thomas [23] concerning E_0-ergodicity and actions of Property (τ) groups. The E_0-ergodicity of Bernoulli actions of Γ_S, Δ_S will be proved below in Section 5.

The use of a measure classification theorem to prove that the image of a mapping into an induced space concentrates on finitely many factors goes back to Adams [1]. In this paper we will require the measure classification theorems [23, 7.3] and [21, 5.7], each of which is based on work of Margulis-Tomanov [16] and Witte-Morris [25]. It must be noted, however, that these measure classification results show only that one of *two* cases obtains: roughly speaking, that either the measure in question concentrates on a suitably small set (the desired case), or else is "evenly distributed." As in [23] and [21], we shall have to eliminate the second possibility, and this amounts to showing that the Bernoulli systems $\Gamma_S \curvearrowright (Y,\nu)^{\Gamma_S}$, $\Delta_S \curvearrowright (Y,\nu)^{\Delta_S}$ do not possess factors of a particular form. For this task we shall require the notion of the *entropy* of a dynamical system, which we will use to restrict the possible factors of Bernoulli actions. This material, including a brief introduction to entropy for the non-expert, will be treated below in Section 6.

Finally, in order to apply Zimmer's theorem we will have to verify that the space induced from a Bernoulli action of Γ_S (Δ_S) is *irreducible.* We introduce the notions of induced space and induced action below in Section 4, and use the notions of strong and mild mixing to establish the irreducibility of the induced spaces that will appear in our argument. We assume, however, that the reader is familiar with the basic machinery of Zimmer's superrigidity theory, and give only a cursory introduction to the notions of Borel cocycle and induced action. For details on this material see Zimmer [27], Adams-Kechris [2], or Adams [1].

In Section 2, we will recall some basic notions and results concerning Borel equivalence relations, ergodic theory, Borel cocycles, and irreducible lattices in semisimple Lie groups. In Section 3 we formally state our main results, and in Sections 7 and 8 we sketch their proofs.

Acknowledgements. I would like to thank Simon Thomas, Alex Furman, and Samuel Coskey for very helpful discussions concerning the material in this paper.

2. Preliminaries

In this section we recall some basic notions and results concerning Borel equivalence relations, ergodic theory, Borel cocycles, and irreducible lattices in semisimple Lie groups.

2.1. **Borel equivalence relations.** Let X be a *standard Borel space*, i.e., a Polish space equipped with its σ-algebra of Borel sets. A *Borel equivalence relation* on X is an equivalence relation $E \subseteq X^2$ that is Borel as a subset of X^2; E is said to be *countable* iff each E-class is countable.

If E and F are Borel equivalence relations on the standard Borel spaces X and Y, respectively, then a *Borel homomorphism* from E to F is a Borel function $f : X \to Y$ such that xEy implies $f(x)Ff(y)$ for all $x, y \in X$. A Borel homomorphism for which xEy iff $f(x)Ff(y)$ is called a *Borel reduction.* If there is a Borel reduction from E to F then we say that E is *Borel reducible* to F, and write $E \leq_B F$. We say that E and F are *Borel bireducible*, and write $E \sim_B F$, iff both $E \leq_B F$ and $F \leq_B E$.

The Borel equivalence relations we shall consider in this paper arise from group actions as follows. Let G be a *lcsc* group, i.e., a locally compact second countable group. Then a *standard Borel G-space* is a standard Borel space X together with a Borel action $(g, x) \mapsto g \cdot x$ of G on X. The corresponding G-orbit equivalence relation on X, denoted E^X_G, is a Borel equivalence relation which by Kechris [12] is bireducible with a countable Borel equivalence relation. Conversely, by Feldman-Moore [5], if E is an arbitrary countable Borel equivalence relation on the standard Borel space X, then there exists a countable group G and a Borel action of G on X such that $E = E_G^X$. For a detailed development of the general theory of countable Borel equivalence relations see Jackson-Kechris-Louveau [10].

2.2. **Ergodic theory.** Suppose that G is a *lcsc* group and X is a standard Borel G-space. A Borel measure μ on X is called *G-invariant* iff $\mu(A) = \mu(gA)$ for all $g \in G$ and Borel $A \subseteq X$. (In this paper, all measures on Polish spaces will be Borel measures, i.e., measures defined on the σ-algebra of Borel sets). A standard Borel G-space (X, G, μ) with invariant Borel measure μ will sometimes be called a *standard Borel system.*

If μ is a G-invariant measure on X, then the action of G on X is said to be *ergodic* iff every G-invariant Borel subset of X is μ-null or μ-conull. The following characterization of ergodicity is well-known.

Proposition 2.1. *Let G be a* lcsc *group, X a standard Borel G-space with G-invariant Borel measure μ. Then the following are equivalent:*

1. *the action of G on (X, μ) is ergodic;*
2. *for any standard Borel space Y and G-invariant Borel function $f : X \to Y$, f is μ-a.e. constant.*
3. *every G-invariant Borel function $f : X \to [0, 1]$ is μ-a.e. constant.*

If X and Y are standard Borel spaces, $f : X \to Y$ is a Borel function, and μ is a probability measure on X, then the measure $f_*\mu$ on Y defined by

$$(f_*\mu)(A) = \mu(f^{-1}(A))$$

for all Borel $A \subseteq Y$ is a probability measure on Y, called the *image* of μ under f.

If E is a countable Borel equivalence relation on a standard Borel space X and μ is a Borel measure on X, then μ is said to be *E-invariant* iff μ is H-invariant for some (equivalently every) countable group H and Borel action of H on X such that $E = E_H^X$. If E, F are Borel equivalence relations on the standard Borel spaces X, Y, respectively, and μ is an E-invariant Borel measure on X, then (E, μ) (or simply E, if μ is understood) is said to be *F-ergodic* iff for any Borel homomorphism $f : X \to Y$

from E to F, there exists a μ-conull subset of X that f maps into a single F-class. It is easily checked that if $F \sim_B F'$, then E is F-ergodic iff E is F'-ergodic, and that if E is F-ergodic and $F' \leq_B F$, then E is F'-ergodic. In light of 2.1, ergodicity is equivalent to $\Delta(Y)$-ergodicity, where $\Delta(Y)$ is the identity relation on any uncountable standard Borel space Y.

Throughout this paper, E_0 will denote the *Vitali equivalence relation* defined on $2^{\mathbb{N}}$ by $\alpha \, E_0 \, \beta$ iff $\alpha(n) = \beta(n)$ for all but finitely many n. If X is a standard Borel G-space with invariant Borel measure μ and E_G^X is E_0-ergodic, then the action of G on X is easily seen to be ergodic.

Finally, we will need the following important definitions.

Definition 2.2. *For each $i = 1, 2$, let Γ_i be a countable group and let X_i be a standard Borel Γ_i-space with an invariant ergodic probability measure μ_i. Then (X_2, Γ_2, μ_2) is a* factor *of (X_1, Γ_1, μ_1) iff there exist*

1. *a group isomorphism $\varphi : \Gamma_1 \to \Gamma_2$ and*
2. *a Borel function $f : X_1 \to X_2$*

such that the following conditions are satisfied:

1. *$f_* \mu_1 = \mu_2$; and*
2. *for each $\gamma \in \Gamma_1$, $f(\gamma \cdot x) = \varphi(\gamma) \cdot f(x)$ for μ_1-a.e. $x \in X_1$.*

If there is a Γ_1-invariant, μ_1-conull subset $X_0 \subseteq X_1$ on which f is a Borel isomorphism, then we call (X_i, Γ_i, μ_i) isomorphic*.*

2.3. **Borel cocycles.** Let G and H be *lcsc* groups, X a standard Borel G-space with invariant probability measure μ. A *cocycle* of the G-space X into H is a function $\alpha : G \times X \to H$ such that

$$\forall g, h \in G \quad \alpha(hg, x) = \alpha(h, gx)\, \alpha(g, x) \quad \mu\text{-a.e.}(x).$$

We say that α is *strict* iff this equation holds for all $x \in X$. If $\beta : G \times X \to H$ is another cocyle into H, we say that α and β are *equivalent*, and write $\alpha \sim \beta$, iff there is a Borel map $b : X \to H$ such that

$$\forall g \in G \quad \beta(g, x) = b(gx)\, \alpha(g, x)\, b(x)^{-1} \quad \mu\text{-a.e.}(x).$$

If Y is a free standard Borel H-space and $f : X \to Y$ is a Borel homomorphism from E^X_G to E^Y_H, then the map $\alpha : G \times X \to H$ defined by $\alpha(g, x)\, f(x) = f(gx)$ is a (strict) Borel cocycle, hereafter referred to as the cocylce corresponding to, or arising from, f. In this case if α is equivalent to a cocycle $\beta : G \times X \to H$ with $\beta(g, x) = b(gx)\, \alpha(g, x)\, b(x)^{-1}$, then the function $f' : X \to Y$ defined by $f'(x) = b(x) \cdot f(x)$ is a Borel homomorphism from E^X_G to E^Y_H that is a Borel reduction iff f is. We often call f' an "adjustment" of f.

2.4. **Lattices and semisimple Lie groups.** Let p be a prime. Throughout this paper, $\mathbb{Q}_p$ will denote the field of p-adic numbers, and $\mathbb{Z}_p$ the ring of p-adic integers. If $G(\mathbb{Q}_p) \leq GL_n(\mathbb{Q}_p)$ is an algebraic $\mathbb{Q}_p$-group, then $G(\mathbb{Q}_p)$ is a *lcsc* group with respect to the *Hausdorff topology*, i.e., the topology obtained by restricting the natural topology on $\mathbb{Q}_p^{n^2}$ to $G(\mathbb{Q}_p)$. More generally, if S is a finite nonempty set of primes, then

$$G = \prod_{p \in S} G(\mathbb{Q}_p)$$

is a *lcsc* group in the product topology. Any topological notions concerning groups of this form will always refer to this topology.

Let G be a *lcsc* group with left Haar measure μ, and recall that any other Haar measure ν on G differs from μ by a scalar multiple. For each $x \in G$, the measure μ_x defined by $\mu_x(E) = \mu(Ex)$ is again a left Haar measure on G, and hence there is $\Delta(x) \in \mathbb{R}^+$ such that $\mu_x = \Delta(x)\mu$. The function $x \mapsto \Delta(x)$ is called the *modular function* of G, and is easily seen to be independent of the choice of μ. It is well known that Δ is a continuous homomorphism from G into the multiplicative group of positive reals. Evidently every left Haar measure on G is also right-invariant iff Δ is identically 1; in this case G is called *unimodular*, and we speak simply of (bi-invariant) Haar measure. Clearly every abelian group is unimodular; compact groups and groups G for which $G/\bar{G}'$ is finite are also unimodular, where here $\bar{G}'$ is the closure of the commutator subgroup of G. In particular, every semisimple Lie group is unimodular.

If G is unimodular and $H \leq G$ is a closed subgroup, then G/H admits a G-invariant Borel measure. If G/H admits a *finite* G-invariant measure then H is said to have *finite covolume* in G; in this case there exists a unique G-invariant probability measure on G/H. A discrete subgroup $\Gamma \leq G$ of finite covolume is called a *lattice* in G.

Let G be a connected semisimple Lie group with finite center, and $\Gamma \leq G$ a lattice. Then Γ is called *irreducible* iff for every non-central normal subgroup N of G, Γ is dense when projected onto G/N. In a closely related notion, an ergodic G-space X with finite invariant measure is called *irreducible* iff for every non-central normal subgroup N of G, the restricted N-action on X is still ergodic. Evidentally a lattice $\Gamma \leq G$ is irreducible iff the action of G on G/Γ is irreducible.

3. Bernoulli Actions of Low Rank Lattices

In this section we fix some terminology and notation for use throughout the rest of the paper, and state our main results.

Throughout (Y, ν), possibly with subscripts, will denote a nontrivial (and possibly countable or finite) standard Borel probability space. By "nontrivial" we mean simply that the Borel probability measure ν does not concentrate on a single point.

If Γ is any countable (discrete) group, we denote by $(Y, \nu)^\Gamma = (Y^\Gamma, \nu^\Gamma)$ the product of identical copies of (Y, ν) indexed by Γ, so that Y^Γ is the space of functions $x : \Gamma \to Y$ which we think of as sequences (x_γ) in Y, and ν^Γ is the product measure on the product σ-algebra of Y^Γ. We may then define a left action of Γ on $(Y, \nu)^\Gamma$ by

$$(\gamma_0 \cdot x)(\gamma) = x(\gamma_0^{-1}\gamma) \quad \text{for } \gamma_0 \in \Gamma,\ x \in Y^\Gamma.$$

We call any such action a *Bernoulli action* of Γ. If $\Gamma \cong \mathbb{Z}$ then we call $\Gamma \curvearrowright (Y, \nu)^\Gamma$ a *Bernoulli automorphism.* Notice the special case $(Y, \nu) = \{0, 1\}$ with the coin-tossing measure. Here the Bernoulli action $\Gamma \curvearrowright 2^\Gamma$ is the shift action of Γ on its power set.

We often call a Borel action of a countable discrete group Γ on a standard Borel probability space *Bernoulli* iff it is isomorphic (in the sense of Definition 2.2) to a Bernoulli action of Γ.

More generally, if I is any countable discrete Γ-set, then writing $\alpha : \Gamma \times I \to I$ for the Γ-action on I, we may define an action of Γ on $(Y, \nu)^I$ by

$$(g_0 \cdot x)(i) = x(\alpha(g^{-1}, i)).$$

We call any such action $\Gamma \curvearrowright (Y, \nu)^I$ a *generalized Bernoulli action* of Γ.

It is well known that if $\Gamma \curvearrowright (Y, \nu)^\Gamma$ is a Bernoulli action, then the subset of Y^Γ on which Γ acts freely is ν^Γ-conull (for instance, see [13, 2.4]). For any Bernoulli action $\Gamma \curvearrowright (Y, \nu)^\Gamma$, we denote this (clearly invariant) subset of Y^Γ by $(Y)^\Gamma$. Frequently we will write $X = (Y)^\Gamma$ and $\mu = \nu^\Gamma \restriction_{(Y)^\Gamma}$, and by abuse of terminology speak of the "free Bernoulli action" $\Gamma \curvearrowright (X, \mu)$.

Throughout this paper the letters S and T, with or without subscripts, will denote finite, nonempty sets of rational primes. We write $S^+ = S \cup \{\infty\}$. For $S = \{p_1, \ldots, p_s\}$, $\mathcal{O}_S$ will denote the ring of integers inside the number field

$$\mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_s}),$$

and $\mathbb{Z}[S]$ will denote the ring $\mathbb{Z}[\frac{1}{p_1}, \ldots, \frac{1}{p_s}]$ generated over $\mathbb{Z}$ and inside $\mathbb{Q}$ by the reciprocals of the primes in S. We then define

$$\Gamma_S = PSL_2(\mathcal{O}_S) \quad \text{and} \quad \Delta_S = PSL_2(\mathbb{Z}[S]).$$

For each $A \subseteq S$, let

$$\sigma_A^S : \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_s}) \hookrightarrow \mathbb{R}$$

be the field embedding that maps $\sqrt{p_i} \mapsto -\sqrt{p_i}$ if $p_i \in A$, $\sqrt{p_i} \mapsto \sqrt{p_i}$ if $p_i \in S \setminus A$, and is the identity on $\mathbb{Q}$, so that in particular $\sigma_\emptyset^S$ is the inclusion embedding. Then, identifying as usual $\mathcal{P}(S) = 2^S$, define

$$\sigma^S : \Gamma_S \to \prod_{2^S} PSL_2(\mathbb{R}) \quad \text{by} \quad \sigma^S(\gamma) = \langle \gamma^{\sigma_A^S} \rangle_{A \subseteq S},$$

where $\gamma^{\sigma_A^S}$ has the obvious meaning and 2^S is ordered, merely for definiteness, lexicographically on the natural ordering for S itself.

Then if we identify Γ_S with its image under σ^S, by Margulis [15, IX(1.7v)],

$$\Gamma_S \leq G_S := \prod_{2^S} PSL_2(\mathbb{R})$$

is an irreducible, noncocompact, arithmetic lattice in the higher rank connected semisimple Lie group G_S.

Furthermore, for each set of primes S let

$$\mathcal{G}_S = \prod_{p \in S} PSL_2(\mathbb{Q}_p),$$

and let

$$\tau^S : \Delta_S \to \mathcal{G}_{S^+} = \prod_{p \in S^+} PSL_2(\mathbb{Q}_p)$$

be the natural diagonal embedding (where as usual we have $\mathbb{Q}_\infty = \mathbb{R}$). Then by Margulis [15, IX(1.7iii)], $\Delta_S \leq \mathcal{G}_{S^+}$ is an irreducible, noncocompact, arithmetic lattice in the higher rank connected semisimple Lie group $\mathcal{G}_{S^+}$. Throughout we shall identify $\sigma^S(\Gamma_S)$, $\tau^S(\Delta_S)$ with Γ_S, Δ_S, and rely upon context to distinguish them.

We are now ready to state our main theorems.

Theorem 3.1. *For $i = 1, 2$, let (Y_i, ν_i) be a non-trivial standard Borel probability space, and let S_i be a finite, nonempty set of primes. Let $X_i \subseteq Y_i^{\Gamma_{S_i}}$ be the subset of $Y_i^{\Gamma_{S_i}}$ on which Γ_{S_i} acts freely as a group of Bernoulli shifts. Let E_i be the $\Gamma_{S_i} \curvearrowright X_i$ orbit equivalence relation. Then $E_1 \leq_B E_2$ implies $S_1 \subseteq S_2$. In particular, if each of $S_1 \setminus S_2$ and $S_2 \setminus S_1$ is nonempty, then E_1 and E_2 are incomparable with respect to Borel reducibility.*

Theorem 3.2. *Theorem 3.1 remains true if Γ_{S_i} is replaced by Δ_{Si}.*

We remark that since the state space (Y_i, ν_i) is allowed to vary in the statements of Theorems 3.1 and 3.2, there is no reason to think that $S_1 \subseteq S_2$ will imply $E_1 \leq_B E_2$. However, if we fix a single state space (Y, ν) and let X_i be the free part of the Bernoulli action of Γ_{S_i} on $Y^{\Gamma_{S_i}}$, then we may prove that $S_1 \subseteq S_2$ implies $E_{\Gamma_{S_1}}^{X_1} \leq_B E_{\Gamma_{S_2}}^{X_2}$ just as in Thomas [24, 3.3]. Hence we obtain the following.

Corollary 3.3. *Let (Y, ν) be a nontrivial standard Borel probability space, and E_S the orbit equivalence relation arising from the restriction of the Bernoulli action $\Gamma_S \curvearrowright (Y, \nu)^{\Gamma_S}$ to the subset of Y^{Γ_S} on which Γ_S acts freely. If T is any other nonempty set* *of primes, then* $E_S \leq_B E_T$ *if and only if* $S \subseteq T$*. The same is true if we replace* Γ_S *with* Δ_S*.*

Proof. If $S \subseteq T$, then Γ_S naturally embeds into Γ_T, and hence we may proceed exactly as in [24, 3.3]. Specifically, fix some point $y_0 \in Y$, and for each $\alpha \in (Y)^{\Gamma_S}$ define a corresponding element $\alpha^* \in Y^{\Gamma_T}$ by

$$\alpha^*(\gamma) = \begin{cases} \alpha(\gamma) & \text{if } \gamma \in \Gamma_S \\ y_0 & \text{otherwise} \end{cases}.$$

It is easily shown that $\alpha^* \in (Y)^{\Gamma_T}$ and that the assignment $\alpha \mapsto \alpha^*$ is a Borel reduction from E_S to E_T. □

4. Irreducibility of Spaces Induced from Bernoulli Actions

In this section we prove a result that will be needed in order to apply Zimmer's cocycle superrigidity theorem [27, 5.2.5] to cocycles associated with Bernoulli actions of Γ_S and Δ_S. We assume that the reader is familiar with basic definitions and facts involving induced spaces and induced actions.

Let (Y, ν) be a nontrivial standard Borel probability space, $(X, \mu) \subseteq (Y, \nu)^{\Gamma_S}$ the full measure, invariant subset on which Γ_S acts freely as a group of Bernoulli shifts, so that $\mu = \nu^{\Gamma_S} \restriction_X$.

We know that Γ_S is an irreducible lattice in the connected semisimple Lie group

$$G_S = \prod_{2^S} PSL_2(\mathbb{R}).$$

Fix a Borel transversal T for G_S/Γ_S containing 1, let m be normalized Haar measure on T (which we identify, as usual, with G_S/Γ_S), and let

$$(\hat{X}, \hat{\mu}) = (X \times T,\, \mu \times m)$$

be the ergodic G_S-space induced from the action of Γ_S on X. One of the key hypotheses of Zimmer's theorem is that the G_S-space $\hat{X}$ is *irreducible*, i.e., that (non-central) normal subgroups of G_S still act ergodically on $\hat{X}$. In this section we will show that this is indeed the case.

In our proof we shall make use of the notions of strong and mild mixing. If G is a countably infinite group and X is a standard Borel G-space with a G-invariant probability measure μ, then the action of G on (X, μ) is said to be *strongly mixing* iff for any two Borel subsets $A, B \subseteq X$, if $\langle g_n \mid n \geq 0 \rangle$ is a sequence of distinct elements of G, then

$$\mu(g_n A \cap B) \to \mu(A)\mu(B) \quad \text{as } n \to \infty.$$

It is well known that Bernoulli actions are strongly mixing (for instance, see [13, 2.3]).

With G, X, and μ as in the previous paragraph, the action of G on (X, μ) is said to be *mildly mixing* (see [8]) iff for every sequence $\langle g_n \mid n \geq 0 \rangle$ of distinct elements of G and for every Borel set $A \subseteq X$ with $0 < \mu(A) < 1$,

$$\liminf_n \mu(A \bigtriangleup g_n A) > 0.$$

It is clear that strong mixing implies mild mixing. Indeed, suppose $G \curvearrowright X$ is strongly mixing, let $\langle g_n \mid n \geq 0 \rangle$ be a sequence of distinct elements of G, and let $A \subseteq X$ be a Borel set that is neither null nor conull. Then

$$\liminf_n \mu(A \bigtriangleup g_n A) = (2\mu(A)(1 - \mu(A))) > 0.$$

Below we shall use the fact [8, 1.1] that if $G \curvearrowright X$ is mildly mixing, and if $G \curvearrowright Y$ is any other properly ergodic G-action on a standard Borel probability space (Y, ν), then the product action of G on $X \times Y$ is also ergodic.

We are now ready to prove Lemma 4.1. Our argument is based in part on the proof of [26, 2.4].

Lemma 4.1. *$(\hat{X}, \hat{\mu})$ is an irreducible G_S-space.*

Proof. Recall that $PSL_2(\mathbb{R})$ is simple, and so the only normal subgroups of G_S are products of full factors $PSL_2(\mathbb{R})$ in G_S. Thus since ergodicity of a subgroup passes upwards, it will suffice to prove that a single factor $N = PSL_2(\mathbb{R}) \leq G_S$ acts ergodically on $(\hat{X}, \hat{\mu})$.

Now, by irreducibility of Γ_S in G_S, Γ_S is ergodic on G_S/N (acting by translations). Furthermore, Γ_S is strongly mixing, and hence mildly mixing, on X. Thus by [8, 1.1] it follows that the product action of Γ_S on $X \times G_S/N$ is ergodic. We show now that this implies ergodicity of N on $(\hat{X}, \hat{\mu})$.

Let ϖ be a G_S-invariant Borel measure on G_S/N, and consider the (product) G_S-space

$$(\hat{X} \times G_S/N,\ \hat{\mu} \times \varpi) = ((X \times G_S/\Gamma_S) \times G_S/N,\ \hat{\mu} \times \varpi).$$

By Zimmer [27, 2.2.2], the ergodicity of N on $(\hat{X}, \hat{\mu})$ is equivalent to the ergodicity of G_S on $(\hat{X} \times G_S/N,\ \hat{\mu} \times \varpi)$. Let Z be any standard Borel probability space, and let

$$F : (X \times G_S/\Gamma_S) \times G_S/N \to Z$$

be an G_S-invariant Borel function. We will show that F is $\hat{\mu} \times \varpi$-a.e. constant. Let

$$\pi : (X \times G_S/\Gamma_S) \times G_S/N \to G_S/\Gamma_S$$

be the canonical projection onto $G_S/\Gamma_S = T$. Then for each $t \in T$, $\pi^{-1}(t)$ is a $t\Gamma_S t^{-1}$-invariant subset of $\hat{X} \times G_S/N$ on which $t\Gamma_S t^{-1}$ acts ergodically. (Each $t\Gamma_S t^{-1}$-action on $\pi^{-1}(t)$ is isomorphic to the ergodic Γ_S-action on $\pi^{-1}(1)$). By Proposition 2.1 and the G_S-invariance of F, it follows that F is $(\mu \times \varpi)$-a.e. constant on each of the fibers $\pi^{-1}(t)$. It then follows from the transitivity of G_S on G_S/Γ_S that the G_S-invariant function F is $(\hat{\mu} \times \varpi)$-a.e. constant on $\hat{X} \times G_S/N$. Again using Proposition 2.1, we conclude that G_S is ergodic on $\hat{X} \times G_S/N$, and the desired result follows. □

Now, of course, we will need the same result for the $\mathcal{G}_S$-space induced from a Bernoulli Δ_S-action when we prove Theorem 3.2. Fortunately the proof just given for Lemma 4.1 goes through without substantial revision.

Specifically, for (Y, ν) a nontrivial standard Borel probability space, define $(X', \mu') \subseteq (Y, \nu)^{\Delta_S}$ to be the free part of the Bernoulli Δ_S-action, as above. We have that Δ_S is an irreducible lattice in the semisimple p-adic Lie group

$$\mathcal{G}_S = \prod_{p \in S^+} PSL_2(\mathbb{Q}_p).$$

Fix a Borel transversal T' for $\mathcal{G}_S/\Delta_S$ and define the induced $\mathcal{G}_S$-space $(\hat{X}', \hat{\mu}')$ as above.

Lemma 4.2. *$(\hat{X}', \hat{\mu}')$ is an irreducible $\mathcal{G}_S$-space.*

Proof. It is easily checked that the proof given for Lemma 4.1 goes through with Γ_S, G_S replaced by Δ_S, $\mathcal{G}_S$, and where this time the normal subgroup N is one of the (simple) factors $PSL_2(\mathbb{Q}_p)$. $\square$

5. E_0-Ergodicity of Orbit Equivalence Relations Arising from Bernoulli Actions

Recall that in order to apply Zimmer's cocycle superrigidity theorems [27] to a cocycle α, we must first verify that α is not equivalent to a cocycle taking values in a proper algebraic subgroup of its target group. In [2], Adams and Kechris exploited the incompatibility of actions of Kazhdan and amenable groups [2, 3.1] in order to verify this hypothesis of Zimmer's theorem. In our case the target group will be either $PSL_2(\mathbb{R})$ or $PSL_2(\mathbb{Q}_p)$ for some prime p, and it is well known that any proper algebraic subgroup of one of these groups is amenable; hence we could apply the techniques of Adams-Kechris [2] if Γ_S and Δ_S were Kazhdan. Of course, it is well known that Γ_S and Δ_S are not Kazhdan. (In fact by Zimmer [28], $PSL_2(\mathbb{C})$ does not contain any finitely generated subgroups with the Kazhdan property).

However, in [23] Thomas noticed that the incompatibility between Kazhdan and amenable groups is to a large extent captured by the notion of E_0-ergodicity. Specifically, Thomas proved [23, Section 4] that if E is an E_0-ergodic countable Borel equivalence relation on a standard Borel space X, and if F is the orbit equivalence relation arising from a Borel action of a (not necessarily countable) *lcsc* amenable group on a standard Borel space Y, then E is F-ergodic. We will be able to use this fact below to verify that Zimmer's theorems do indeed apply to our cocycles, provided that Bernoulli actions of Γ_S and Δ_S are E_0-ergodic. We proceed to show this now.

Lemma 5.1. *Let (Y,ν) be a nontrivial standard Borel probability space, and let Γ be any nonamenable countable discrete group. Let $X \subseteq Y^\Gamma$ be the (invariant, conull) subset on which Γ acts freely as a group of Bernoulli shifts. Then $\Gamma \curvearrowright X$ is E_0-ergodic.*

Proof. We shall make use of the notion of "almost invariant sets," which in the ergodic setting is closely related to E_0-ergodicity by a result of Jones and Schmidt [11]. Recall that an action of a countable group G on a measure space (Ω, ω) by measure-preserving transformations *has almost invariant sets* iff there is a sequence $\{A_n\}$ of measurable sets with measures bounded away from 0 and 1 such that for all $g \in G$,

$$\omega(g \cdot A_n \triangle A_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Write F for the orbit equivalence relation arising from the Bernoulli action $\Gamma \curvearrowright (Y, \nu)^\Gamma$. Since this action is ergodic (in fact strongly mixing [13, 2.3]), it follows by Jones-Schmidt [11] that F is E_0-ergodic if and only if $\Gamma \curvearrowright (Y, \nu)^\Gamma$ does not have almost invariant sets. (For a clear statement and proof of this result, see Hjorth-Kechris [9, A2.2]). And by Kechris and Tsankov [13, 1.2], a Bernoulli action of a countably infinite group G has almost invariant sets if and only if G is amenable. Hence (F, ν^Γ) is E_0-ergodic. As any Borel homomorphism $F : X \to 2^{\mathbb{N}}$ from E_Γ^X to E_0 can be trivially extended to a Borel homomorphism $\tilde{F} : Y^\Gamma \to 2^{\mathbb{N}}$ from F to E_0, the desired result follows. $\square$

Remark 5.2. Since Γ_S and Δ_S contain nonabelian free subgroups and hence are not amenable, it follows that free Bernoulli actions of these groups are E_0-ergodic.

6. Entropy and Factors of Bernoulli Shifts

In this section we will use the notion of the *Kolmogorov-Sinai entropy* of a dynamical system to show, loosely speaking, that in many cases Bernoulli systems do not have linear algebraic factors. This notion of entropy was initially defined for discrete-time measure preserving dynamical systems in [14]. For convenience we recall the definition and some basic facts below.

Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space, with $T : X \to X$ a Borel, measure-preserving transformation of X, so that $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{B}$. Given a finite partition $\xi = \{C_1, \ldots, C_n\}$ of X into Borel subsets, define the *entropy of the partition* ξ to be the number

$$H(\xi) = -\sum_{C_i \in \xi} \mu(C_i) \log \mu(C_i),$$

where $\mu(C_i)\log\mu(C_i) = 0$ for C_i null. Next define the *entropy of the dynamical system* $(X, \mathcal{B}, \mu, T)$ *with respect to the partition* ξ to be

$$h(T, \xi) = \lim_{n\to\infty} \frac{1}{n} H(\xi \vee T^{-1}\xi \vee \cdots \vee T^{-n+1}\xi),$$

where the join $\vee_i \xi_i = \xi_1 \vee \cdots \vee \xi_n$ of a finite sequence of finite partitions ξ_i is simply the collection of all intersections of sets in $\bigcup_i \xi_i$. (For the existence of this limit, see [6, 4.4]). Finally, define the *Kolmogorov-Sinai entropy* of $(X, \mathcal{B}, \mu, T)$ to be

$$h(T) = \sup_{\xi} h(T, \xi),$$

where the supremem is taken over all finite partitions ξ of X into Borel sets.

Notice that if T is invertible with measurable inverse, then T induces a $\mathbb{Z}$-action on X defined by $n \cdot x = T^n(x)$. Generalizing this, we may view a measurable, measure-preserving $\mathbb{Z}^d$-action on a measure space $(X, \mathcal{M}, \mu)$ as a multi-parameter discrete-time dynamical system, and attempt to extend the definition of entropy to this setting. (See [22] for a survey of the rich theory of the dynamics of algebraic $\mathbb{Z}^d$-actions). In fact, it turns out that one may generalize Kolmogorov-Sinai entropy to measurable, measure-preserving actions of a large class of amenable groups; for the most general results see Ornstein and Weiss [18]. For clear accounts of the entropy of one-parameter dynamical systems see [3, I.3] or [6].

It is easy to check that if the dynamical systems $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{N}, \nu, S)$ are isomorphic, then they have the same entropy. What is quite remarkable, however, is Ornstein's celebrated result that in the context of Bernoulli actions, entropy is a *complete* isomorphism invariant. Again, the most general results can be found in [18], but we shall need only the following:

Proposition 6.1 (Ornstein). *For* $i = 1, 2$, *let* (Y_i, ν_i) *be nontrivial standard Borel probability spaces, and* $\mathbb{Z} \curvearrowright (Y_i, \nu_i)^{\mathbb{Z}}$ *Bernoulli automorphisms. Then* $(Y_i^{\mathbb{Z}}, \mathbb{Z}, \nu_i^{\mathbb{Z}})$ *are isomorphic if and only if they have the same entropy.*

We shall also need the following result of Ornstein's, characterizing factors of Bernoulli automorphisms. (See Ornstein [17], and also the Corollary and discussion following Theorem 4.6 in Chapter I.3 of [3]).

Proposition 6.2 (Ornstein). *Let (Y, ν) be a nontrivial standard Borel probability space, and $\mathbb{Z} \curvearrowright (Y, \nu)^{\mathbb{Z}}$ a Bernoulli automorphism. Then any factor of $\mathbb{Z} \curvearrowright (Y, \nu)^{\mathbb{Z}}$ is also Bernoulli, i.e., is isomorphic to a Bernoulli automorphism.*

And finally, we shall need the following easy result, which gives a condition for a generalized Bernoulli action to be Bernoulli.

Proposition 6.3. *Suppose Γ is a countable discrete group acting on a countable discrete index set I, and suppose $\Gamma \curvearrowright (Y, \nu)^I$ is the generalized $(\Gamma \curvearrowright I)$-Bernoulli Γ-action on $(Y, \nu)^I$. If the action of Γ on I is free, then $\Gamma \curvearrowright (Y, \nu)^I$ is isomorphic to the Bernoulli action $\Gamma \curvearrowright ((Y, \nu)^{I/\Gamma})^{\Gamma}$. In particular, $\Gamma \curvearrowright (Y, \nu)^I$ is Bernoulli.*

Proof. Let $T \subseteq I$ be a transversal for the orbit equivalence relation of the Γ-action on I, and identify T with I/Γ. Define a function

$$f : Y^I \to (Y^{I/\Gamma})^{\Gamma} \qquad \text{by} \qquad f(x)(\gamma)(t) = x(\gamma t).$$

It is easily checked that f gives the desired isomorphism. □

We are now ready to establish the results needed in the proof of Theorems 3.1, 3.2.

Lemma 6.4. *Suppose Γ is a countable discrete group. Let (Y, ν) be a nontrivial standard Borel probability space, and I a countable discrete index set on which Γ acts freely. Let $\Gamma \curvearrowright (Y, \nu)^I$ be the generalized $(\Gamma \curvearrowright I)$-Bernoulli action. Suppose $(Z, \varphi(\Gamma), \zeta)$ is a nontrivial factor of $\Gamma \curvearrowright (Y, \nu)^I$. If $\gamma \in \Gamma$ has infinite order, then $\varphi(\gamma)$ acts on (Z, ζ) with positive Kolmogorov-Sinai entropy.*

Proof. Suppose $\varphi : \Gamma \to \Lambda$ is a Borel group isomorphism, with (Z, ζ) a nontrivial standard Borel Λ-space, and let $F : Y^I \to Z$ be a Borel function such that

1. $F_* \nu^I = \zeta$, and
2. for all $\gamma \in \Gamma$, $F(\gamma x) = \varphi(\gamma) F(x)$ for ν^I-a.e. $x \in Y^I$.

Suppose $\gamma \in \Gamma$ has infinite order, and consider the restriction of the generalized Bernoulli Γ-action on $(Y, \nu)^I$ to the cyclic subgroup $\mathbb{Z} \cong \langle \gamma \rangle \leq \Gamma$. By Proposition 6.3, this restricted action

$$\langle \gamma \rangle \curvearrowright (Y, \nu)^I$$

is isomorphic to the Bernoulli $\langle\gamma\rangle$-action

$$\langle\gamma\rangle \curvearrowright \left((Y,\nu)^{I/\langle\gamma\rangle}\right)^{\langle\gamma\rangle}.$$

Pushing forward through (F,φ), and using Proposition 6.2, we have that

$$\langle\varphi(\gamma)\rangle \curvearrowright (Z,\zeta)$$

is Bernoulli. That is, $\langle\varphi(\gamma)\rangle \curvearrowright (Z,\zeta)$ is isomorphic to a Bernoulli automorphism, say $\mathbb{Z} \curvearrowright (Z_0,\zeta_0)^{\mathbb{Z}}$, where (Z_0,ζ_0) must be nontrivial since (Z,ζ) is nontrivial. But this implies that the Kolmogorov-Sinai entropy of $\mathbb{Z} \curvearrowright (Z_0,\zeta_0)^{\mathbb{Z}}$ is nonzero (for instance, see [3, Example 3 on page 42]), and as entropy is an isomorphism invariant for Bernoulli actions, it follows that the Kolmogorov-Sinai entropy of

$$\langle\varphi(\gamma)\rangle \curvearrowright (Z,\zeta)$$

is nonzero. This completes the proof of the lemma. □

Corollary 6.5. *Let (Y,ν) be a nontrivial standard Borel probability space, $X \subseteq Y^{\Gamma_S}$ the (invariant, conull) subset on which Γ_S acts freely as a group of Bernoulli shifts. Suppose $\varphi : G_S \to H$ is an isomorphism of algebraic groups. Suppose $M \leq H$ is a closed subgroup with finite covolume, with m (H-invariant) normalized Haar measure on H/M. Then $(H/M, \varphi(\Gamma_S), m)$ is not a factor of $(X, \Gamma_S, \nu^{\Gamma_S} \restriction_X)$.*

Proof. Suppose for contradiction that $(H/M, \varphi(\Gamma_S), m)$ is a factor of $(X, \Gamma_S, \nu^{\Gamma_S} \restriction_X)$. Then $(H/M, \varphi(\Gamma_S), m)$ is also a factor of $(Y^{\Gamma_S}, \Gamma_S, \nu^{\Gamma_S})$. Fix a torsion-free unipotent element $\gamma \in \Gamma_S$. By Proposition 6.4, $\varphi(\gamma)$ acts on $(H/M, m)$ with positive entropy. But by [4, Appendix] and the unipotence of $\varphi(\gamma)$, the translation action of $\varphi(\gamma)$ on $(H/M, m)$ has zero entropy, a contradiction. □

An identical proof yields the following, which we will need to prove Theorem 3.2.

Corollary 6.6. *Let (Y,ν) be a nontrivial standard Borel probability space, $X \subseteq Y^{\Delta_S}$ the (invariant, conull) subset on which Δ_S acts freely as a group of Bernoulli shifts. Suppose $\varphi : \mathcal{G}_S \to H$ is an isomorphism of algebraic groups. Suppose $M \leq H$ is a closed subgroup with finite covolume, with m (H-invariant) normalized Haar measure on H/M. Then $(H/M, \varphi(\Delta_S), m)$ is not a factor of $(X, \Delta_S, \nu^{\Delta_S} \restriction_X)$.*

7. Proof of Theorem 3.1

In this section we will sketch the proof of Theorem 3.1. Our proof will follow substantially the proof of [21, 3.1], so we will concentrate on the differences between the two proofs and direct the reader to Schneider [21] for the details.

Suppose that $f : X_1 \to X_2$ is a Borel reduction from E_1 to E_2, and let

$$\alpha : \Gamma_{S_1} \times X_1 \to \Gamma_{S_2}$$

be the cocycle corresponding to f, so that

$$\alpha(\gamma, x) \cdot f(x) = f(\gamma x)$$

for all $\gamma \in \Gamma_{S_1}$ and $x \in X_1$. Recall that for $i = 1, 2$, $(X_i, \mu_i) \subseteq (Y_i, \nu_i)^{\Gamma_{S_i}}$ is the $\nu_i^{\Gamma_{S_i}}$-conull subset on which Γ_{S_i} acts freely as a group of Bernoulli shifts; in particular, we write $\mu_i = \nu_i^{\Gamma_{S_i}} \restriction_{X_i}$. Let m_i be normalized Haar measure on G_{S_i}/Γ_{S_i}, and let

$$(\hat{X}_i \hat{\mu}_i) = (X_i \times G_{S_i}/\Gamma_{S_i},\ \mu_i \times m_i)$$

be the ergodic G_{S_i}-space induced from the action of Γ_{S_i} on X_i. Let $\hat{\alpha} : G_{S_1} \times X_1 \to \Gamma_{S_2}$ be the cocycle induced from α. By Lemma 4.1, $\hat{X}_1$ is an irreducible G_{S_1}-space.

Now let $\sigma^{S_2} : \Gamma_{S_2} \hookrightarrow G_{S_2}$ be the embedding which realizes Γ_{S_2} as an irreducible lattice in G_{S_2}, and for each $A \subseteq S_2$ let $\pi_A^{S_2} : G_{S_2} \to PSL_2(\mathbb{R})$ be the canonical projection onto the factor of G_{S_2} corresponding to A. Consider the cocycles

$$\hat{\alpha}_A = \pi_A^{S_2} \circ \sigma^{S_2} \circ \hat{\alpha} : G_{S_1} \times \hat{X}_1 \to PSL_2(\mathbb{R}).$$

Then since $\Gamma_{S_1} \curvearrowright X_1$ is E_0-ergodic by Lemma 5.1, we may use Lemmas [21, 5.3] and [21, 5.4] exactly as in [21] to set up the application of Zimmer's cocycle superrigidity theorem [27, 5.2.5] to each of the cocycles $\hat{\alpha}_A$. Applying Zimmer yields group homomorphisms $\psi_A : G_{S_1} \to PSL_2(\mathbb{R})$ to which the $\hat{\alpha}_A$ are equivalent.

Notice that at this point in the proof of [21, Theorem 3.1], all of the subsequent adjustments and cocycle prestidigitations involve only the groups $\Gamma_{S_1} \leq G_{S_1}$, $\Gamma_{S_2} \leq G_{S_2}$, and do not depend in any manner on the spaces X_1, X_2. Thus from the original Borel reduction $f : X_1 \to X_2$ from E_1 to E_2 with corresponding cocycle $\alpha : \Gamma_{S_1} \times X_1 \to$

Γ_{S_2}, we obtain a new Borel reduction $\tilde{f} : X_1 \to \hat{X}_2$ corresponding to the cocycle $\beta : \Gamma_{S_1} \times X_1 \to G_{S_2}$ defined by $\beta(\gamma, x) = \varphi(\gamma)$, where

$$\varphi : G_{S_1} \to G_{S_2}$$

is an injective group homomorphism such that for all $\gamma \in \Gamma_{S_1}$, each component of $\varphi(\gamma)$ is either $\gamma^{\sigma_A^{S_1}}$, or $\gamma^{\sigma_A^{S_1}}$ with main diagonal scaled by -1, for some Galois automorphism $\sigma_A^{S_1}$. The Borel reduction $\tilde{f}$ and the homomorphism φ satisfy

$$\tilde{f}(\gamma x) = \varphi(\gamma)\tilde{f}(x) \quad \text{for all } \gamma \in \Gamma_{S_1} \text{ and for } \mu_1\text{-a.e. } x \in X_1,$$

giving us the familiar picture illustrated below in Figure 1.

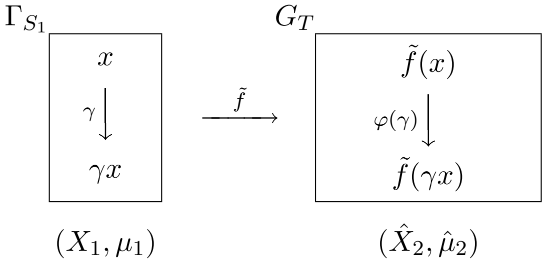

FIGURE 1

As in [21], our problem now is that $\tilde{f}$ takes values in $\hat{X}_2$ instead of in X_2. Fortunately, since $(X_1, \Gamma_{S_1}, \mu_1)$ has no factors isomorphic to $(G_{S_1}/M, \Gamma_{S_1}, m)$, with $M \leq G_{S_1}$ closed of finite covolume and m normalized Haar, we may proceed as in the proof of [21, 3.1]. Thus let $\eta : \hat{X}_2 \to G_{S_2}/\Gamma_{S_2}$ be the canonical projection, and let $\omega = (\eta \circ \tilde{f})_* \mu_1$. Then ω is a $\varphi(\Gamma_{S_1})$-invariant, $\varphi(\Gamma_{S_1})$-ergodic probability measure on G_{S_2}/Γ_{S_2}, and, by applying Lemma 6.5 in place of [21, 5.11] at the appropriate point in the proof of [21, 3.1], we see that ω is supported on a finite set $\Omega_0 \subseteq G_{S_2}/\Gamma_{S_2}$.

As this is the point in the proof that differs most substantially from the corresponding point in the proof of [21, Theorem 3.1], we provide a few more details. We wish to prove, as a analogue to [21, Lemma 6.3], that ω is finitely supported. As in the proof of [21, 6.3], if we write $H = \varphi(G_{S_1})$ and $C = \mathrm{Stab}_H(\omega)$, so that ω is supported on a single C-orbit by [21, Lemma 5.7], this amounts to showing that $C \neq H$. Thus for contradiction assume $C = H$, so that ω is supported on the single H-orbit Ω.

As H is transitive on Ω, there exists a proper closed subgroup M of H such that (Ω, ω) and $(H/M, m)$ are isomorphic as H-spaces, where m is Haar probability measure on H/M. Of course, this implies that (Ω, ω) and $(H/M, m)$ are also isomorphic as $\varphi(\Gamma_{S_1})$-spaces. In particular, $(H/M, m)$ is a φ-factor of the Γ_{S_1}-space (X_1, μ_1), contradicting Corollary 6.5.

As Ω_0 is finite and $\varphi(\Gamma_{S_1})$-invariant, there is a finite index subgroup $\Gamma^0_{S_1}$ of Γ_{S_1} whose image under φ acts trivially on Ω_0. As in the proof of [21, 3.1], we may untwist φ by an element $u\Gamma_{S_2} \in \Omega_0$ to obtain a conjugate $\overline{\varphi}$ of φ such that $\overline{\varphi}(\Gamma^0_{S_1}) \leq \Gamma_{S_2}$. At this point, since an element of $PSL_2(\mathbb{R})$ conjugates a finite index subgroup of Γ_{S_1} into Γ_{S_2}, we obtain $S_1 \subseteq S_2$ exactly as in [21, Proposition 6.5]. This completes the proof of Theorem 3.1.

8. Proof of Theorem 3.2

In this final section we will briefly sketch the proof of Theorem 3.2. Since a proof of 3.2 amounts to little more than combining the techniques of Thomas [23] with those used above to prove Theorem 3.1, we shall be terse, and refer the reader to Thomas [23] for many details.

As above, suppose that $f : X_1 \to X_2$ is a Borel reduction from E_1 to E_2, and let $\alpha : \Delta_{S_1} \times X_1 \to \Delta_{S_2}$ be the cocycle corresponding to f. Let

$$(\hat{X}_i,\, \hat{\mu}_i) = (X_i \times \mathcal{G}_{S_i^+}/\Delta_{S_i},\ \mu_i \times m_i)$$

be the ergodic $\mathcal{G}_{S_i^+}$-space induced from the action of Δ_{S_i} on X_i, and let $\hat{\alpha} : \mathcal{G}_{S_1^+} \times X_1 \to \Delta_{S_2}$ be the cocycle induced from α. By Lemma 4.2, $\hat{X}_1$ is an irreducible $\mathcal{G}_{S_1^+}$-space.

Now let $\sigma^{S_2} : \Delta_{S_2} \hookrightarrow \mathcal{G}_{S_2^+}$ be the embedding which realizes Δ_{S_2} as an irreducible lattice in $\mathcal{G}_{S_2^+}$, and for each $p \in S_2^+$ let $\pi_p^{S_2^+} : G_{S_2^+} \to PSL_2(\mathbb{Q}_p)$ be the canonical projection. Consider the cocycles

$$\hat{\alpha}_p = \pi_p^{S_2^+} \circ \sigma^{S_2^+} \circ \hat{\alpha} : G_{S_1^+} \times \hat{X}_1 \to PSL_2(\mathbb{Q}_p).$$

Then since $\Delta_{S_1} \curvearrowright X_1$ is E_0-ergodic by Lemma 5.1, we may use Lemmas [23, 8.4] and [23, 8.5] exactly as in Thomas [23] to set up the application of Zimmer's cocycle superrigidity Theorem [27, 10.1.6] to each of the cocycles $\hat{\alpha}_p$.

After applying Zimmer and arguing exactly as in Thomas [23], we obtain a Borel function $\overline{f} : X_1 \to \hat{X}_2$ and a Borel cocycle $\beta : \Delta_{S_1} \times X_1 \to \mathcal{G}_{S_2^+}$ such that for all $\gamma \in \Delta_{S_1}$ and for μ_1-a.e. $x \in X_1$,

$$\overline{f}(\gamma \cdot x) = \beta(\gamma, x) \cdot \overline{f}(x).$$

Finally, as in Thomas, we study the distribution of $\overline{f}(X_1)$ within $\hat{X}_2$. In fact, one may easily check that the proof of [23, 8.7] goes through exactly as in [23] provided only that we substitute Corollary 6.6 in place of [23, 6.4]. From [23, 8.7], then, we may conclude that $S_1 \subseteq S_2$, as desired.

References

[1] S. Adams, *Containment does not imply Borel reducibility*, in: Set Theory: The Jahnal Conference (Ed: S. Thomas), DIMACS Series, vol. 58, American Mathematical Society, 2002, pp. 1-23.

[2] S. Adams and A. S. Keckris, *Linear algebraic groups and countable Borel equivalence relations*, J. Amer. Math. Soc. **13** (2000), 909-943.

[3] L. A. Bunimovich, et. al., *Dynamical Systems, Ergodic Theory and Applications* (Ed. Ya. G. Sinai), Encyclopedia of Math. Sci., **100**, Springer, 2000.

[4] S. G. Dani, *Spectrum of an affine transformation*, Duke Math. J., **44**(1) (1977), 129–155.

[5] J. Feldman and C. C. Moore, *Ergodic equivalence relations and Von Neumann algebras I*, Trans. Amer. Math. Soc. **234** (1977), 289-324.

[6] M. Foreman, *A descriptive view of ergodic theory*, in: Descriptive Set Theory and Dynamical Systems (Ed. M. Foreman, et. al.), London Math. Soc. Lecture Note Series **277**, Cambridge University Press (2000), 87-173.

[7] A. Furman, *On Popa's Cocycle Superrigidity Theorem*, Inter. Math. Res. Notieces, **2007** (2007), 1-46.

[8] J. Hawkins and C. Silva, *Characterizing mildly mixing actions by orbit equivalence of products*, New York J. Math. **3A** (1998), 99-115.

[9] G. Hjorth and A. S. Kechris, *Rigidity theorems for actions of product groups and countable Borel equivalence relations*, Mem. Amer. Math. Soc. **177**, no. 833 (2005).

[10] S. Jackson, A. S. Kechris, and A. Louveau, *Countable Borel equivalence relations*, J. Math. Logic, **2** (2002), 1-80.

[11] V. Jones and K. Schmidt, *Asymptotically invariant sequences and approximate finiteness*, Amer. J. Math., **109** (1987), 91-114.

[12] A. S. Kechris, *Countable sections for locally compact group actions*, Ergodic Theory Dynamical Systems, **12** (1992), 283-295.

[13] A. Kechris and T. Tsankov, *Amenable actions and almost invariant sets*, Proc. of the Amer. Math. Soc., **136 (2)** (Fubruary 2008), 687-697.

[14] A. N. Kolmogorov, *A new metric invariant of transient dynamical systems and automorphisms of Lebesgue spaces*, Dokl. Adad. Nauk SSSR, Ser. Math., **119**, 861-864 (1958).

[15] G. A. Margulis, *Discrete Subgroups of Semisimple Lie Groups*, Erg. der Math. und ihrer Grenz. **17**, Springer-Verlag, 1991.

[16] G. A. Margulis and G. M. Tomanov, *Measure rigidity for almost linear groups and its applications*, Journal d'Analyse Math., **69** (1996), 25-54.

[17] D. Ornstein, *Factors of Bernoulli shifts are Bernoulli shifts*, Adv. in Math. **5** (1971), 349-364.

[18] D. Ornstein and B. Weiss, *Entropy and isomorphism theorems for actions of amenable groups*, J. d'Analyse Math. 48 (1987), 1-141.

[19] S. Popa, *Cocycle and orbit equivalence superrigidity for malleable actions of ω-rigid groups*, Inventiones Mathematicae, **170 (2)** (2007), 243-295.

[20] S. Popa, *On the superrigidity of malleable actions with spectral gap*, J. Amer. Math. Soc. **21** (2008), 981–1000.

[21] S. Schneider, *Borel incomparability results for $SL_2(\mathcal{O})$-actions*, preprint.

[22] K. Schmidt, *Dynamical Systems of Algebraic Origin*, Birkhäuser Verlag, 1995.

[23] S. Thomas, *Property (τ) and countable Borel equivalence relations*, J. Math. Logic **7** (2007), 1-34.

[24] S. Thomas, *Popa superrigidity and countable Borel equivalence relations*, accepted for publication in Ann. Pure Appl. Logic as of 7/31/07.

[25] D. Witte, *Measurable quotients of unipotent translations on homogeneous spaces*, Trans. Amer. Math. Soc. **345** (1994), 577-594.

[26] R. J. Zimmer, *Ergodic actions of semisimple groups and product relations*, Annals of Mathematics, **118** (1983), 9-19.

[27] R. J. Zimmer, *Ergodic Theory and Semisimple Groups*, Birkhäuser, 1984.

[28] R. J. Zimmer, *Kazhdan groups acting on compact manifolds*, Invent. Math. **75** (1984), 425-436.