Journal of Mathematical Sciences, Vol. 137, No. 2, 2006

A BRIEF ACCOUNT OF THE SCIENTIFIC AND PEDAGOGICAL WORK OF YU. V. LINNIK

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Yurii Vladimirovich Linnik was born on January 8, 1915 in the town Belaya Tserkov'; he died on June 30, 1972 in Leningrad.

The parents of Yu. V. Linnik, Vladimir Pavlovich Linnik and Mariya Abramovna Linnik, were teachers.

The broad areas of scientific interest of Yu. V. Linnik lied in number theory, probability theory, and mathematical statistics.

The scientific work of Yu. V. Linnik in number theory can be divided into several periods in each of which he was studying some definite problems.

His first youthful interest was generated by a problem concerning the representation of an integer by a ternary quadratic form of a certain type. This interest was stimulated by Professor B. A. Venkov who delivered lectures on number theory at Leningrad State University at that time and whose scientific work was related to this area, going back to Gauss.

A ternary form can be treated as an ellipsoid in the three-dimensional space, and the problem on the number of representations of an integer is reduced to the computation of the number of integer points on an ellipsoid. By that time, this problem had been solved for a sphere. Linnik succeeded in solving this problem for a larger class of ternary forms.

Later he had shown an increasing interest in Hardy–Littlewood's circle method and in the method of trigonometric sums originated by I. M. Vinogradov.

In 1940, he began intensively studying I. M. Vinogradov's conjecture on the least nonresidue, which was a great favorite of his and on which he had been working until his death in 1972. In 1941, he published a note in *Doklady Akademii Nauk SSSR*, where he proved that Vinogradov's conjecture is valid for all moduli, except maybe for a bounded set of them. That note was entitled "Large sieve", and its future was splendid. The problems stated in it gave rise to a whole scientific direction, and at present it is a leading direction in number theory, which offered hundreds of papers and tens of books all over the world. The ideas of that paper enabled one to prove an arithmetic analog of the extended Riemann hypothesis (ERH) in the middle 1960s, and since an immense circle of problems in number theory is related to the ERH, it is clear why the future of that little note was so extraordinary.

In the years of the Second World War, Yu. V. Linnik published a series of profound papers devoted to I. M. Vinogradov's method, in which he found connections between this method and the p-adic arithmetic of local fields, and he obtained an estimate of a polynomial Weyl sum with lowering power $1/n^2 \log n$, where n is the degree of the polynomial. Later, these ideas were developed in papers of A. A. Karatsuba.

At the same time, Yurii Vladimirovich studied the theory of Dirichlet L-series and compared the L-series method and I. M. Vinogradov's method. He tried to state the problem of obtaining I. M. Vinogradov's estimate for sums over prime numbers with the help of density theorems for L-series. This setting of the problem turned out to be very fruitful. It enabled one to develop new methods in multiplicative and additive number theory. In 1944, the search had led him to a famous theorem on the least prime number of an arithmetic progression. In a series of these investigations, he created a new density method in the d-aspect, which was further developed intensively in our country and abroad. Many papers appeared in which the authors tried to lower Linnik's constant (the exponent of a power of d, which is greater than or equal to the minimal prime number).

That theorem of Yu. V. Linnik is indicative of a fruitful influence of I. M. Vinogradov's method on multiplicative number theory. But in 1946, Yurii Vladimirovich realized that there are two-way connections in these areas.

By that time, he succeeded in proving that I. M. Vinogradov's estimate for prime numbers can indeed be obtained by the method of density theorems. This enabled him to give a new proof for a famous Vinogradov–

Translated from *Zapiski Nauchnykh Seminarov POMI*, Vol. 322, 2005, pp. 5–9.

1072-3374/06/1372-4609

Goldbach theorem on three primes. Thus he enriched the methods of additive number theory with the help of ideas borrowed from multiplicative number theory.

In that series of papers, Yu. V. Linnik developed the density method of L-series in the t-aspect. Note that the t- and d-aspects, elaborated by Linnik in the theory of L-series, are different in nature. The first aspect concerns the analytic nature of Dirichlet L-functions, because t is the imaginary part of the analytic parameter s. The second aspect relates to the arithmetic nature of L-series, because d is the modulus of the Dirichlet character. These two aspects were being successfully developed by many authors in parallel, until, finally, they were united early in the 1970s into one method, the so-called "large sieve" in the interpretation of H. Montgomery.

It should be noted that in 1943 Linnik found his famous elementary solution of Waring's problem, which is widely known owing to the book "Three pearls in number theory" by A. Ya. Khinchin.

In the late 1940 and early 1950s, Linnik was deeply thinking about the binary Goldbach problem. He found that the circle method combined with the ERH cannot solve this problem. Not much was needed, i.e., a sequence of logarithmic density. These reflections gave rise to a series of papers. We note two of these papers, which summed up the work on this theme: "Some conditional theorems concerning the binary Goldbach problem" (1952) and "Addition to prime numbers of powers of one and the same number" (1953).

In the middle 50s, his youthful interest in ternary forms was regenerated. He continued developing his pre-war ideas. The main distinction of these investigations from the pre-war ones is in their ergodic nature. He had studied not only the total number of integer points on a whole sphere but their distribution on separate parts of it. In that series of papers, he succeeded in proving a fundamental result, namely, the uniform distribution of integer points on a sphere (jointly with A. V. Malyshev). The concluding paper of that series is "Asymptotic-geometrical and ergodic properties of sets of integer points an a sphere" (1957). At the same time, he was studying the problem concerning the distribution of integer points on hyperboloids. This problem is closely related to arithmetic binary definite quadratic forms. If we take a class of such forms with a fixed discriminant and treat the coefficients of these forms as independent parameters, then we obtain a hyperboloid. If we restrict ourselves to reduced forms, then the Gauss reduction conditions cut out a noncompact domain of reduction on the hyperboloid. The number of integer points in this domain coincides with the number of different forms with one and the same discriminant. A natural question arises: How many integer points are there in some part of the domain of reduction? This is an analog of the problem on the number of integer points on parts of a sphere. In the case of a sphere, we may consider it in the ordinary Euclidean space, but in the case of a hyperboloid the Euclidean metric does not work. It is necessary to embed the hyperboloid into a non-Euclidean space and to use the Lobachevskii metrics. The idea of passage to a non-Euclidean metric was suggested by B. A. Venkov in 1951. Yu. V. Linnik developed and augmented it significantly. Note that, in this direction, there is a crucial defference between the cases of two-sheeted and one-sheeted hyperboloids. The first case is associated with positive definite binary quadratic forms or, in other words, with imaginary quadratic fiedls. The second case corresponds to indeterminate forms or real quadratic fields. The crucial difference is due to the presence of the unity in the real field, which has an infinite order and which is absent in the imaginary fields. Yurii Vladimirovich investigated ergodic properties of a two-sheeted hyperboloid, and his pupil B. F. Skubenko studied a one-sheeted hyperboloid. We note the main paper of that series: "Asymptotic distribution of reduced binary quadratic forms in connection with the Lobachevskii geometry" (1955).

In the late 1950s and early 1960s, Linnik originated a new, dispersion method in number theory. In 1960, he solved by this method a known Hardy–Littlewood problem on the representation of each integer number as the sum of a prime number and two squares. The essence of this method is as follows. Most commonly, we cannot calculate the asymptotics of an arithmetic function in a progression with large modulus. But sometimes the possibility exists of calculating the dispersion of this sum over all large moduli. Yurii Vladimirovich discovered that this is sufficient for many problems and, in particular, for the Hardy–Littlewood problem. For this reason, the method was named *dispersion method*, although in essence there is no connection with probability methods. The ideas of this method were presented in a series of papers. A comprehensive idea of this method can be gained in reading his monograph "Dispersion method in binary additive problems" (1961). Later he repeatedly turned to this method, developing and amplifying it.

In the middle 1960s, Linnik turned to ternary forms for the third time. His brief note in *Doklady Akademii Nauk SSSR* entitled "Hyperelliptic curves and the least prime quadratic residue" (1966) had given impetus to this turn. In this note, he proved that such a residue is of order of the 4th root of the modulus. This simplified significantly the whole theory of ternary forms developed by Linnik in the previous two periods. At that time, the lack of information about small prime residues affected the method adversely and made it more complicated in technique, because it was necessary to “bypass” large residues. After the note of 1966, these difficultes have disappeared. The theory became clear and beautiful. Yurii Vladimirovich described it in the monograph entitled “Ergodic properties of algebraic fields” (1967). To that time he had discovered ergodic properties of the dispersion method. The combination of geometry and the dispersion method gave rise to a series of papers written jointly with B. M. Bredikhin. The main paper in this series is “Asymptotics and ergodic properties of solutions of the generalized Hardy–Littlewood equation”, where these ideas are described most fully.

Later on, Linnik realized that he can obtain an elementary proof of the Vinogradov–Goldbach theorem on three primes. This was presented in his last paper on number theory, entitled “A new method in analytic number theory” written jointly with B. M. Bredikhin. This paper appeared in 1974, after the death of Yu. V. Linnik, in the book “Urgent problems in analytic number theory”.

Now it is clear that the number-theoretic papers of Linnik have determined to a large extent contemporary number theory, especially, in the part concerning the L-series and density methods.

Translated by N. B. Lebedinskaya.