Solution to Prob. 29(a) in Section 1.6

Start with the hint given in the book. Let B be the basis for $W_1 \cap W_2$, and let B_1 be the basis obtained for W_1 by extending B and let B_2 the corresponding extension to a basis for W_2. Let $B_3 = B_1 \cup B_2$. We have to show that B_3 is a linearly independent collection. To this end, suppose that a linear combination of elements in B_3 is zero:

$$\sum_{i=1}^{k} a_i u_i + \sum_{j=1}^{m} b_j v_j + \sum_{l=1}^{p} c_j w_j = 0.$$

Let us call the first sum x the second y and the third z. We thus have $x+y+z=0$. Moreover, we know that $x \in W_1 \cap W_2$, $y \in W_1$ and $z \in W_2$. Now we have $y = -x-z$ and therefore $y \in W_2$ as well because both x and z are in W_2. Hence $y \in W_1 \cap W_2$. Since B_1 is a basis for the intersection, we must have $y = \sum_{i=1}^{k} \alpha_i u_i$. But this means that

$$\sum_{j=1}^{m} b_j v_j - \sum_{i=1}^{k} \alpha_i u_i = 0$$

, which because of linear independence of B_1 implies that $b_j = 0$ for all $j = 1, \ldots, m$ and $\alpha_i = 0$ for all $i = 1, \ldots, k$. In particular, $y = 0$, which implies $x + z = 0$, and now by the linear independence of B_2 we have that $a_i = 0$ for $i = 1, \ldots, k$ and $c_l = 0$ for $l = 0, \ldots, p$. So we have proved the linear independence of B_3.

Proving B_3 is a generating set is easier and I leave it to you to do for yourself!