Solution to Prob. 29(a) in Section 1.6

Start with the hint given in the book. Let $B$ be the basis for $W_1 \cap W_2$, and let $B_1$ be the basis obtained for $W_1$ by extending $B$ and let $B_2$ the corresponding extension to a basis for $W_2$. Let $B_3 = B_1 \cup B_2$. We have to show that $B_3$ is a linearly independent collection. To this end, suppose that a linear combination of elements in $B_3$ is zero:

$$\sum_{i=1}^{k} a_i u_i + \sum_{j=1}^{m} b_j v_j + \sum_{l=1}^{p} c_l w_l = 0.$$

Let us call the first sum $x$ the second $y$ and the third $z$. We thus have $x + y + z = 0$. Moreover, we know that $x \in W_1 \cap W_2$, $y \in W_1$ and $z \in W_2$. Now we have $y = -x - z$ and therefore $y \in W_2$ as well because both $x$ and $z$ are in $W_2$. Hence $y \in W_1 \cap W_2$. Since $B_1$ is a basis for the intersection, we must have $y = \sum_{i=1}^{k} \alpha_i u_i$. But this means that

$$\sum_{j=1}^{m} b_j v_j - \sum_{i=1}^{k} \alpha_i u_i = 0,$$

which because of linear independence of $B_1$ implies that $b_j = 0$ for all $j = 1, \ldots, m$ and $\alpha_i = 0$ for all $i = 1, \ldots, k$. In particular, $y = 0$, which implies $x + z = 0$, and now by the linear independence of $B_2$ we have that $a_i = 0$ for $i = 1, \ldots, k$ and $c_l = 0$ for $l = 0, \ldots, p$. So we have proved the linear independence of $B_3$.

Proving $B_3$ is a generating set is easier and I leave it to you to do for yourself!