Solutions, Assignment #3

We need to relate the matrices A, A_ϵ, and $f_i(\epsilon)$. To the equations describing the propogation of error, in Chapter 8.5.

First we notice that $\|A - A_\epsilon\| = |\epsilon|$. This is easily seen, since the matrix $A - A_\epsilon$ only has one non-zero entry, so all the row sums are zero except for the row that ϵ is in.

Next we need to relate our $f_i(\epsilon)$ to matrix norms involving error. We notice that:

$$|f_i(\epsilon)| \leq \|\vec{x} - \vec{x}_\epsilon\|$$

Because:

$$\vec{x} - \vec{x}_\epsilon = [f_1(\epsilon), f_2(\epsilon), f_3(\epsilon)]^T$$

And for any i,

$$|f_i(\epsilon)| \leq \max_j |f_j(\epsilon)| = \|\vec{x} - \vec{x}_\epsilon\|$$

Now we do a little calculating, and get $\|A\| = \max\{4, 5, 14\} = 14$.

$$A^{-1} = \begin{bmatrix} 36/49 & -13/49 & -4/49 \\ 9/49 & 9/49 & -1/49 \\ -5/49 & -5/49 & 6/49 \end{bmatrix}$$

So we get

$$\begin{aligned} \|A\| &= 14 \\ \left\|A^{-1}\right\| &= \max\{54/49, 19/49, 16/49\} = 54/49 \\ \text{cond}(A) &= \|A\| \left\|A^{-1}\right\| = 108/7 \end{aligned}$$

and

$$\|x\| = max\{1, 1, 1, \} = 1$$

So the last thing we need is the inequality relating error in A to error in $\vec{x}$. From the bottom of page 253:

$$\begin{aligned} \frac{\|\vec{x} - \vec{x}_\epsilon\|}{\|x\|} &\leq \frac{\text{cond}(A)}{1 - \text{cond}(A)\frac{\|A - A_\epsilon\|}{\|A\|}} \frac{\|A - A_\epsilon\|}{\|A\|} \\ \frac{\frac{|f_i(\epsilon)|}{\|x\|}}{\|x\|} &\leq \frac{\|\vec{x} - \vec{x}_\epsilon\|}{\|x\|} \leq \frac{108/7}{1 - 108/7\frac{\epsilon}{14}} \frac{\epsilon}{14} \\ |f_i(\epsilon)| &\leq \frac{\epsilon \cdot 108/7}{14 - \epsilon \cdot 108/7} \end{aligned}$$

This puts a bound on how large $|f_i(\epsilon)|$ can be, in relation to ϵ. We can see this in the graph, because it tells us that our plot of ϵ vs. $f_i(\epsilon)$ cannot be too steep.