Homework 2

Due Date: November 10, 2011.

1. Let $f(c)$ equal the $c$-color Ramsey number $R(3,3,\ldots,3)$ (namely, the smallest integer $n$ such that any $c$-coloring of the edges of $K_n$ contains a monochromatic triangle).

Show that $(f(c_1 + c_2) - 1) \geq (f(c_1) - 1) \cdot (f(c_2) - 1)$. Thus show that $f(c) \geq 5^{\lfloor c/2 \rfloor}$.

Recall that the proof of the Ramsey theorem shows that $f(c) \leq O(c!) = \exp(c \log c)$. Today it is unknown what the correct behavior for $f(c)$ is.

2. Let $p \in (0,1)$ be fixed. We will study the chromatic number of $G(n,p)$.

Recall that $G(n,p)$ has clique number $(2 + o(1)) \cdot \log \frac{1}{1-p} n$. Thus $G(n,p)$ has independence number $(2 + o(1)) \cdot \log \frac{1}{1-p} n$, and so $G(n,p)$ has chromatic number at least

$$(1 + o(1)) \cdot \frac{n}{2 \log \frac{1}{1-p} n}.$$ 

Show that the greedy coloring algorithm colors $G(n,p)$ with $O\left(\frac{n}{\log n}\right)$ colors with high probability.

3. Let $s,t$ be fixed. For which $\alpha \in (0,2)$ is $G(n,n-\alpha)$ almost sure to have no $K_s,t$? For which $\alpha \in (0,2)$ is $G(n,n-\alpha)$ almost sure to have fewer than $n/2$ copies of $K_s,t$?

Use this to prove a lower bound on the Turan number $\text{ex}(n,K_s,t)$. For which $s,t$ does this match the upper bound we saw in class?

4. (a) Let $H = (L,R,E)$ be any bipartite graph with $|L| = |R| = n$ and $|E| = e$. Show that there is a subgraph $H' = (L',R',E')$ of $H$ with at least $e/2n$ edges such that every vertex in $R'$ has degree at least $e/2n$ in $H'$.

(Aside: Must there be a subgraph $H'' = (L'',R'',E'')$ with at least $e/100$ edges such that every vertex in $L''$ and every vertex in $R''$ has degree at least $e/100n$?)

(b) Let $T$ be a tree with $k$ edges. Let $v$ be a vertex of $T$.

Prove, by induction on $T$, that in every bipartite graph $H = (L,R,E)$ with $|L| = |R| = n$ and $|E| = e$,

$$|\{ \varphi \in \text{hom}(T,H) \mid \varphi(v) \in R \}| \geq c_k \left( \frac{e}{n} \right)^k \cdot n,$$

where $c_0 = 1$, and $c_i = c_{i-1} \cdot 2^{-i}$ for each $i > 0$.

Hint: Use $H'$ from part (a).

(c) Show that any graph $G$ with $n$ vertices and edge density $\beta$, we have:

$$\frac{\text{hom}(T,G)}{n^{k+1}} \geq c_k \beta^k.$$
(d) **Sidorenko’s conjecture for trees:** Use the previous statement to show that in any graph $G$ with $n$ vertices and edge density $\beta$, we have:

$$\frac{\text{hom}(T, G)}{n^{k+1}} \geq \beta^k.$$

5. **(Fun with Homomorphisms)**

(a) Write $F \to G$ if $\text{hom}(F, G) > 0$. Give examples of graphs $F, G$ such that $F \to G$ and $G \to F$.

(b) A **core** of a graph $G$ is a graph $F$ such that $F \to G$, $G \to F$, and every homomorphism from $F$ to $F$ is an isomorphism. Show that every graph $G$ has a core.

(c) Suppose $F_1$ and $F_2$ are graphs such that for all graphs $G$, $F_1 \to G$ if and only $F_2 \to G$. What can you say about $F_1$ and $F_2$?

(d) Suppose $G_1$ and $G_2$ are graphs such that for all graphs $F$, $F \to G_1$ if and only if $F \to G_2$. What can you say about $G_1$ and $G_2$?

(e) Suppose $\text{hom}(F_1, G) = \text{hom}(F_2, G)$ for all graphs $G$. What can you say about $F_1$ and $F_2$?

(f) Suppose $\text{hom}(F, G_1) = \text{hom}(F, G_2)$ for all graphs $F$. What can you say about $G_1$ and $G_2$?